

## BOCHNER-RIESZ PROFILE OF ANHARMONIC OSCILLATOR

$$\mathcal{L} = -\frac{d^2}{dx^2} + |x|$$

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**ABSTRACT.** We investigate spectral multipliers, Bochner-Riesz means and convergence of eigenfunction expansion corresponding to the Schrödinger operator with anharmonic potential  $\mathcal{L} = -\frac{d^2}{dx^2} + |x|$ . We show that the Bochner-Riesz profile of the operator  $\mathcal{L}$  completely coincides with such profile of the harmonic oscillator  $\mathcal{H} = -\frac{d^2}{dx^2} + x^2$ . It is especially surprising because the Bochner-Riesz profile for the one-dimensional standard Laplace operator is known to be essentially different and the case of operators  $\mathcal{H}$  and  $\mathcal{L}$  resembles more the profile of multidimensional Laplace operators. Another surprising element of the main obtained result is the fact that the proof is not based on restriction type estimates and instead entirely new perspective have to be developed to obtain the critical exponent for Bochner-Riesz means convergence.

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## 1. INTRODUCTION

One of the most significant and central problems in harmonic analysis is convergence of the Fourier transform and series. This problem leads in a natural way to the question of convergence of Bochner-Riesz means of Fourier integrals and series. In a systematic manner the topic was initiated in the 1930s by Bochner. Since then it has attracted very significant attention. Nevertheless there still remain many fundamental problems to be resolved. Detailed account of the main ideas and development of this area can be found for example in [15, Chapter 8], [33, Section IX.2], [34, Chapter II], [38] or [25].

Using the language of the spectral theory the problem of Convergence of Bochner-Riesz means of Fourier series can be formulated for any eigenfunction expansion of any abstract self-adjoint operators. Convergence and equivalently boundedness of Bochner-Riesz means for general differential operators or varies specific operators were studied among the

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other by Christ, Karadzhov, Koch, Ricci, Seeger, Sogge, Stempak, Tataru, Thagavelu and Zienkiewicz, see [8, 22, 23, 24, 30, 35, 36, 37, 41]. See also [16]. This paper is a continuation of these affords in particular case of the operator  $\mathcal{L} = -\frac{d^2}{dx^2} + |x|$ .

The theory of  $L^p$  spectral multipliers is essentially equivalent to Bochner-Riesz analysis but is more flexible and precise, see discussion in Section 7. Therefore we adopt this approach in this paper and we state our main result Theorem 1.2 below in the language of spectral multipliers. In this context it is worth mentioning that the theory of  $L^p$  spectral multipliers itself also attracts significant interest. Initially spectral theory for self-adjoint operators was motivated by Fourier multiplier type results of Mikhlin and Hörmander [19, 28]. These results restricted to radial Fourier multipliers can be written in terms of spectral multipliers for standard Laplace operators and opened question of possible generalisation to larger class of self-adjoint operators, see also discussion in [7]. In our approach we investigate Mikhlin and Hörmander multipliers together with Bochner-Riesz analysis as essentially the same research area. The literature devoted to the spectral multipliers is much too broad to be listed here so we refer the reader to [10, 5, 11, 16] for large class of examples of papers devoted to this area of harmonic analysis. Some recent developments going in somehow different direction can be found in [26]. A few other interesting examples of spectral multiplier results in various settings can be found in [1, 2, 8, 9, 27, 29, 30, 34, 35, 40].

In [41] Thangavelu showed that the profiles of Bochner-Riesz means convergence for the standard Laplace operator in one dimension and one dimensional harmonic oscillator are essentially different. This indicates that in the theory of spectral multipliers one has to study specific examples of operators because the results can be essential different even if considered ambient spaces have the same topological or homogenous dimension.

In this paper we consider one dimensional Schrödinger type operator with anharmonic potential

$$(1.1) \quad \mathcal{L} = -\frac{d^2}{dx^2} + |x|$$

which can be precisely defined using the standard approach of quadratic forms. It is well-known that this type of operator is self-adjoint and admits a spectral resolution

$$\mathcal{L} = \int_0^\infty \lambda dE_{\mathcal{L}}(\lambda),$$

where the  $E_{\mathcal{L}}(\lambda)$  are spectral projectors. For any bounded Borel function  $F: [0, \infty) \rightarrow \mathbb{C}$ , we define the operator  $F(\mathcal{L})$  by the formula

$$(1.2) \quad F(\mathcal{L}) = \int_0^\infty F(\lambda) dE_{\mathcal{L}}(\lambda).$$

In virtue of spectral theory the operator  $F(\mathcal{L})$  is well defined and bounded on  $L^2(\mathbb{R})$ . The operators  $\mathcal{L} = -\frac{d^2}{dx^2} + |x|$  and  $\mathcal{H} = -\frac{d^2}{dx^2} + x^2$  are examples of Schrödinger operators with potential growing to infinity when  $x$  approaches  $\infty$  or  $-\infty$ . It is well-known that for such operators there exist orthonormal bases of their eigenfunctions. That is there exists a system  $\{h_n\}_{n=1}^\infty$ ,  $h_n \in L^2(\mathbb{R})$  such that  $\mathcal{L}h_n = \lambda_n h_n$  and for any  $f \in L^2(\mathbb{R})$  we have  $\|f\|_2^2 = \sum_{n=1}^\infty |\langle f, h_n \rangle|^2$ . Hence

$$f = \sum_{n=1}^\infty h_n \langle f, h_n \rangle.$$

The convergence in the above sum is understood in sense of  $L^2(\mathbb{R})$ . A classical problem in harmonic analysis is whether this series is also convergent in other  $L^p(\mathbb{R})$  spaces and it is one of important rationale for developing the theory of Bochner-Riesz analysis and more general spectral multipliers. Note that now a spectral multiplier for operator  $\mathcal{L} = -\frac{d^2}{dx^2} + |x|$  given by formula (1.2) can be written as

$$F(\mathcal{L})f = \sum_{n=1}^{\infty} F(\lambda_n) h_n \langle f, h_n \rangle.$$

Spectral multiplier theorems describe sufficient conditions for function  $F$  which guarantee the operator extends to a bounded operator acting on  $L^p$  spaces for some range of  $p$ .

One of more interesting and significant instants of spectral multipliers are Bochner-Riesz means. To define it we set

$$(1.3) \quad \sigma_R^\alpha(\lambda) = \begin{cases} (1 - \lambda/R)^\alpha & \text{for } 0 \leq \lambda \leq R \\ 0 & \text{for other } \lambda. \end{cases}$$

We then define the operator  $\sigma_R^\alpha(\mathcal{L})$  using (1.2). The main problem considered in Bochner-Riesz analysis is to find exponent  $\alpha_{cr}(p)$  such that the operators  $\sigma_R^\alpha(\mathcal{L})$  are bounded uniformly in  $R$  on  $L^p$  for all  $\alpha > \alpha_{cr}(p)$ . Recall that uniform boundedness and convergence of Bochner-Riesz means are equivalent. In addition to our discussion above we refer readers to [4, 21, 33, 38] and references therein for some further detailed background information about Bochner-Riesz analysis and spectral multipliers. We also want to mention that in most of the cases full description of Bochner-Riesz profile of general differential operators or even the standard Laplace operator is an open problem, see [8, 30, 35, 36].

As we mentioned before our study is devoted to Bochner-Riesz means and spectral analysis of particular operator  $\mathcal{L} = -\frac{d^2}{dx^2} + |x|$ . It is motivated by results described in [3, 41], where combination of results obtained by Askey, Wainger and Thangavelu provide full description (except of the endpoints) of convergence of Bochner-Riesz means for the harmonic oscillator  $\mathcal{H} = -\frac{d^2}{dx^2} + x^2$  and it is one of very few examples when such full picture was obtained. Also in the case of the operator  $\mathcal{L}$  which we consider here we obtain a complete description of the critical exponent  $\alpha_{cr}(p)$  for all  $1 \leq p \leq \infty$ .

One of more interesting features of our results is the fact that the range of convergence of Bochner-Riesz means for operator  $\mathcal{L}$  coincides completely with the same range for harmonic oscillator. To be more precise we note that the description of convergence of Bochner-Riesz means which follows from Askey, Wainger and Thangavelu's results and which is stated in [41, Theorem 5.5] can be summarised in the following way.

**Proposition 1.1.** *Consider the operator  $\mathcal{H} = -\frac{d^2}{dx^2} + x^2$ . Then  $\sigma_R^\alpha(\mathcal{H})$  is uniformly bounded on  $L^p$  if the point  $(1/p, \alpha)$  belongs to regions A or B, that is if  $\alpha > \max\{0, \frac{2}{3}|\frac{1}{2} - \frac{1}{p}| - \frac{1}{6}\}$ , see figure 1. Next if  $(1/p, \alpha)$  belongs to regions C, that is if  $\alpha < \max\{0, \frac{2}{3}|\frac{1}{2} - \frac{1}{p}| - \frac{1}{6}\}$ , then  $\sup_{R>0} \|\sigma_R^\alpha(\mathcal{H})\|_{p \rightarrow p} = \infty$ .*

Our main result is stated in Theorem 1.2 below. As we explain above we prefer to formulate our main result in terms of spectral theory and it is stated in the theorem below. Then to be able to compare the Bochner-Riesz profiles of operators  $\mathcal{H}$  and  $\mathcal{L}$  we will formulate corresponding description of Bochner-Riesz convergence for  $\mathcal{L}$  in Theorem 1.3 below.

**Theorem 1.2.** *Suppose that  $\mathcal{L}$  is an anharmonic oscillator defined by (1.1) and that  $\text{supp } F \subset [1/2, 1]$ . Assume next that  $1 \leq p \leq \infty$ ,  $s > \max\{\frac{1}{2}, \frac{2}{3}|\frac{1}{2} - \frac{1}{p}| + \frac{1}{3}\}$  and that  $F \in H^s$ .*

*Then the operators  $F(t\mathcal{L})$  are uniformly bounded on space  $L^p(\mathbb{R})$  and*

$$\sup_{t>0} \|F(t\mathcal{L})\|_{p \rightarrow p} \leq C \|F\|_{H^s} < \infty.$$

The proof of Theorem 1.2 is described in Section 5. However essential preparatory ingredients of the proof are described in Sections 3 and 4. Results discussed in Section 3 are rather standard but non-trivial so we include them for the sake of completeness. In sections 4 and 5 we develop essentially new techniques for handling spectral multiplier operators. These two sections are the most significant and interesting part of the paper. We want to stress again that surprisingly the proof is not based on restriction type estimates as it is the case in most of known results in Bochner-Riesz analysis.

As we mentioned above the following result which is mainly a consequence of Theorem 1.2 gives a complete picture of Bochner-Riesz convergence for the operator  $\mathcal{L}$ .

**Theorem 1.3.** *Suppose that  $\mathcal{L}$  is defined by (1.1) that is  $\mathcal{L} = -\frac{d^2}{dx^2} + |x|$ . Then  $\sigma_R^\alpha(\mathcal{L})$  is uniformly bounded on  $L^p$  if  $\alpha > \max\{0, \frac{2}{3}|\frac{1}{2} - \frac{1}{p}| - \frac{1}{6}\}$ , which means the point  $(1/p, \alpha)$  belongs to regions A or B. Moreover if  $\alpha < \max\{0, \frac{2}{3}|\frac{1}{2} - \frac{1}{p}| - \frac{1}{6}\}$ , this is if  $(1/p, \alpha)$  belongs to regions C, then  $\sup_R \|\sigma_R^\alpha(\mathcal{L})\|_{p \rightarrow p} = \infty$ .*

The positive part of Theorem 1.3 is a rather straightforward consequence of Theorem 1.2 and the implication essentially boils down to the fact that  $\sigma^\alpha \in H^s$  if and only if  $\alpha + 1/2 > s$ . The negative part essentially follows from our discussion in Section 6 and Theorem 6.1 below. We conclude the proof of Theorem 1.3 at the end of Section 6.

**Remark 1.4.** *We want to point out that Theorem 1.3 follows from Theorem 1.2 but Theorem 1.2 is (at least formally) essentially stronger than Theorem 1.3, see the discussion in Section 7. The question whether the operator  $\mathcal{L}$  can be replaced by the harmonic oscillator  $\mathcal{H}$  in the statement of Theorem 1.2 is an open problem.*

**Remark 1.5.** *The endpoint convergence of Bochner-Riesz means for operator  $\mathcal{L}$  remains an open question except for  $p = 4$  and  $p = 4/3$ . That is we do not know if  $\sigma_R^\alpha(\mathcal{L})$  is uniformly bounded on  $L^p$  for the critical exponent  $\alpha = \max\{0, \frac{2}{3}|\frac{1}{2} - \frac{1}{p}| - \frac{1}{6}\}$ . However it follows from the necessary condition described in Section 6 that  $\|\sigma_R^0(\mathcal{L})\|_{4 \rightarrow 4}$  is not uniformly bounded.*

The following picture describes the convergence of Bochner-Riesz means for operators  $\mathcal{L}$  and  $\mathcal{H}$ . Note that the means are convergent in both regions A and B. The range A is common for all abstract operators in dimension 1, for which the corresponding semigroups and heat kernels satisfies Gaussian bounds, see [11]. However the division between the parts B (convergent) and C (divergent) possibly depends on the operator. Indeed in case of the standard Laplace operator on  $\mathbb{R}$  or on one dimensional torus Bochner-Riesz means converge in both regions B and C whereas for considered operators  $\mathcal{L}$  and  $\mathcal{H}$  the means are uniformly bounded only in B and they are not convergent in part C. To sum up the region A is completely understood for all abstract operators in dimension 1 whereas the division between regions B (convergent) and C (divergent) is not known for most operators with exception of the standard Laplace operator, harmonic oscillator  $\mathcal{H}$  and now also an operator  $\mathcal{L}$ .

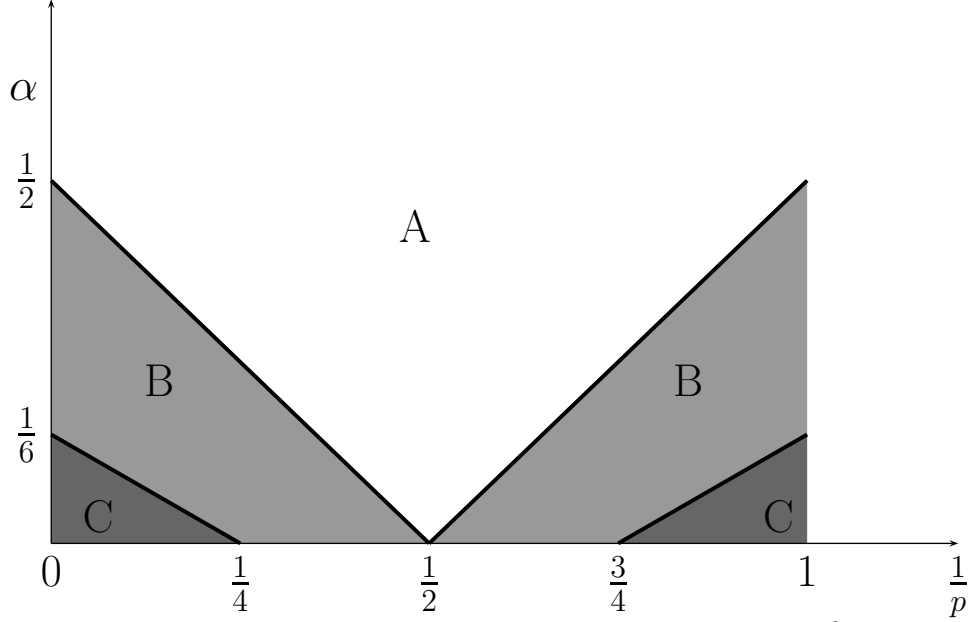


FIGURE 1. Convergence of Bochner-Riesz means for operators  $-\frac{d^2}{dx^2} + |x|$  and harmonic oscillator  $-\frac{d^2}{dx^2} + x^2$ . Note that for both operators the convergence in region A follows from more general results which just required Gaussian upperbounds for the corresponding semigroups, see [11].

Through out of the paper,  $W_s^p$  denotes the Soblev space defined by the norm  $\|F\|_{W_s^p} = \|(Id - \frac{d^2}{dx^2})^{s/2} F\|_{L^p}$ . Next for  $p = 2$  we set  $W_s^p = H^s$ .  $f \sim w$  means that there exist positive constants  $C_1, C_2$  such that  $C_1 w \leq f \leq C_2 w$ .

Plan of the paper. In Section 2 we give basic description of eigenfunction expansion of the operator  $\mathcal{L} = -\frac{d^2}{dx^2} + |x|$  which is based on Airy function. Next in Section 3 we describe some general spectral multiplier results which are required in the proof of the main result. In Section 4 we discuss in details properties of the Airy operator and function. The proof of the main result that is Theorem 1.2 is concluded in Section 5. Next in Section 6 we discuss the necessary condition for convergence of Bochner-Riesz means.

## 2. EIGENFUNCTION EXPANSION OF THE OPERATOR $\mathcal{L}$

We start our discussion with a precise description of the spectral decomposition of the operator  $\mathcal{L}$  based on the results described in [13].

We recall that the Airy function, which we denote by  $\text{Ai}$  is defined as the inverse Fourier transform of the function  $\xi \rightarrow \exp(i\xi^3/3)$ , see [20, Definition 7.6.8, Page 213]. In the sequel we will need the following properties of spectral decomposition of operator  $\mathcal{L}$  which are proved in Section 2 of [13].

**Proposition 2.1.** *Suppose that the operator  $\mathcal{L}$  acting on  $L^2(\mathbb{R})$  is defined by formula (1.1). Then its spectral decomposition satisfies the following properties:*

- (1) *The operator  $\mathcal{L}$  has only a pointwise spectrum and its eigenvalues belong to  $(1, \infty)$ . In particular the first eigenvalue is larger than 1.*

- (2) Every eigenvalue of  $\mathcal{L}$  is simple and the only point of accumulation of the eigenvalue sequence is  $\infty$ .  
 (3) The spectrum of  $\mathcal{L}$  is described by the following formula

$$\{\lambda \in \mathbb{R} : \text{Ai}(-\lambda) = 0 \text{ or } \text{Ai}'(-\lambda) = 0\}.$$

Moreover, the normalized eigenfunction  $\phi_n$  corresponding to the eigenvalue  $\lambda_n$  can be described as

$$(2.1) \quad \phi_n(u) = \begin{cases} A_n \text{Ai}(u - \lambda_n) & \text{for } u \geq 0 \\ (-1)^{n+1} A_n \text{Ai}(-u - \lambda_n) & \text{for } u \leq 0 \end{cases}$$

where

$$(2.2) \quad A_n = \left( 2 \int_{-\lambda_n}^{\infty} |\text{Ai}(u)|^2 du \right)^{-1/2}.$$

- (4) The eigenvalues  $\lambda_n$  behave asymptotically in the following way

$$(2.3) \quad \lim_{n \rightarrow \infty} \lambda_n \left( \frac{3\pi}{4} n \right)^{-2/3} = 1$$

and

$$(2.4) \quad \frac{\pi}{2} \lambda_{n+1}^{-1/2} \leq \lambda_{n+1} - \lambda_n \leq \frac{\pi}{2} \lambda_n^{-1/2}$$

for all  $n = 1, 2, \dots$

*Proof.* Proposition 2.1 is just reformulation of Proposition 2.1, Corollary 2.2, Facts 2.3, 2.7 and Theorem 2.6 of [13]. The complete asymptotic of eigenvalues  $\lambda_n$  including more precise version of relations (2.3) and (2.4) is described in [12].  $\square$

In what follows we also need the following lemma.

**Lemma 2.2.** *Let  $\phi_n$  is the normalized eigenfunction corresponding to the eigenvalue  $\lambda_n$  defined in (2.1). Then*

$$(2.5) \quad |\phi_n(u)| \leq \begin{cases} C \lambda_n^{-\frac{1}{4}} \left( ||u| - \lambda_n| + 1 \right)^{-\frac{1}{4}}, & \text{for } |u| \leq \lambda_n \\ C \lambda_n^{-\frac{1}{4}} \exp \left( -\frac{2}{3} ||u| - \lambda_n|^{\frac{3}{2}} \right), & \text{for } |u| > \lambda_n. \end{cases}$$

In addition

$$(2.6) \quad \|\phi_n\|_{L^p} \sim \begin{cases} \lambda_n^{\frac{1}{p} - \frac{1}{2}}, & \text{for } 1 \leq p < 4, \\ \lambda_n^{-\frac{1}{4}} (\ln \lambda_n)^{\frac{1}{4}}, & \text{for } p = 4, \\ \lambda_n^{-\frac{1}{4}}, & \text{for } p > 4, \end{cases}$$

where  $f \sim w$  means that there exist positive constants  $C_1, C_2$  such that  $C_1 w \leq f \leq C_2 w$ .

*Proof.* It is well known that the Airy function  $\text{Ai}$  is bounded. In the proof, we also need the following estimates for  $\text{Ai}$ :

There exists a constant  $C$  such that for all  $u > 0$

$$(2.7) \quad |\text{Ai}(u)| \leq C \exp(-2u^{\frac{3}{2}}/3) u^{-\frac{1}{4}};$$

In addition for all  $u < 0$  the asymptotic behaviour of the Airy function as  $u$  goes to minus infinity can be described in the following way

$$(2.8) \quad \text{Ai}(u) = (\pi)^{-\frac{1}{2}} |u|^{-\frac{1}{4}} \left( \sin\left(\frac{2}{3}|u|^{\frac{3}{2}} + \frac{\pi}{4}\right) + O(|u|^{-\frac{3}{2}}) \right),$$

[20, (7.6.20) and (7.6.21), Page 215].

Next by (2.2)

$$\begin{aligned} A_n^{-2} &= 2 \int_{-\lambda_n}^{\infty} |\text{Ai}(u)|^2 du \\ &\sim \int_{-\lambda_n}^{-1} |\text{Ai}(u)|^2 du + \int_{-1}^{\infty} |\text{Ai}(u)|^2 du. \end{aligned}$$

The Airy function is smooth and bounded so by (2.7)

$$\int_{-1}^{\infty} |\text{Ai}(u)|^2 du \leq C < \infty$$

and

$$\int_{-\lambda_n}^{-1} |\text{Ai}(u)|^2 du \sim \int_1^{\lambda_n} u^{-\frac{1}{2}} du \sim \lambda_n^{\frac{1}{2}}.$$

Hence  $A_n \sim \lambda_n^{-1/4}$  so (2.5) follows from (2.1), (2.7) and (2.8).

Alternatively note that  $\lambda_n \phi_n(0)^2 + \phi_n'(0)^2 = 1$  so

$$\lambda_n A_n^2 |\text{Ai}(-\lambda_n)|^2 + A_n^2 |\text{Ai}'(-\lambda_n)|^2 = 1.$$

Hence  $A_n \sim \lambda_n^{-1/4}$  by asymptotics (2.8) and similar asymptotics for derivative of the Airy function described in Proposition 4.2 below.

Now by (2.1),

$$\begin{aligned} \|\phi_n\|_{L^p}^p &= \int_0^{\infty} |\phi_n(u)|^p du + \int_{-\infty}^0 |\phi_n(u)|^p du \\ &= 2A_n^p \int_0^{\infty} |\text{Ai}(u - \lambda_n)|^p du \\ (2.9) \quad &= 2A_n^p \left( \int_{-\lambda_n}^{-1} |\text{Ai}(u)|^p du + \int_{-1}^{\infty} |\text{Ai}(u)|^p du \right). \end{aligned}$$

The Airy function is smooth and bounded so by (2.7),

$$(2.10) \quad \int_{-1}^{\infty} |\text{Ai}(u)|^p du \leq C < \infty.$$

Then by (2.8), for  $1 \leq p < 4$

$$\int_{-\lambda_n}^{-1} |\text{Ai}(u)|^p du \sim \int_1^{\lambda_n} u^{-\frac{p}{4}} du \sim \lambda_n^{1-\frac{p}{4}},$$

for  $p = 4$

$$\int_{-\lambda_n}^{-1} |\text{Ai}(u)|^p du \sim \int_1^{\lambda_n} u^{-1} du \sim \ln \lambda_n,$$

and for  $p > 4$

$$\int_{-\lambda_n}^{-1} |\text{Ai}(u)|^p du \sim \int_1^{\lambda_n} u^{-\frac{p}{4}} du \sim C.$$

Now Lemma 2.2 follows from (2.9), (2.10) and the estimates for  $A_n$ .  $\square$

### 3. SPECTRAL MULTIPLIER THEOREMS FOR ABSTRACT SELF-ADJOINT OPERATORS

The aim of this section is to prove two auxiliary results - Lemma 3.1 and Proposition 3.2 which we use in the proofs of Theorems 1.2 and 1.3.

Set  $I_\lambda = [-\lambda, \lambda]$ . Let  $\chi_{I_\lambda}$  be the characteristic function of interval  $I_\lambda$  we denote by  $I_\lambda$  also a projection acting on  $L^p(\mathbb{R})$  spaces defined by

$$I_\lambda f(x) = \chi_{I_\lambda} f(x)$$

for any  $f \in L^p(\mathbb{R})$ . Similarly we set  $I_\lambda^c f(x) = \chi_{I_\lambda^c} f(x) = (1 - \chi_{I_\lambda}) f(x)$ .

We first observe that if  $\text{supp } F \subset [1/2, 1]$  then it is enough to estimate  $F(\mathcal{L}/\lambda)$  on the interval  $I_{2\lambda}$  and that the part of multiplier  $F(\mathcal{L}/\lambda)$  outside  $I_{2\lambda}$  is negligible. More precisely we show that

**Lemma 3.1.** *Suppose that  $\mathcal{L}$  is an anharmonic oscillator defined by (1.1) and that  $F: \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function such that  $\text{supp } F \subset [1/2, 1]$ . Then for any  $1 \leq p \leq \infty$*

$$\|F(\mathcal{L}/\lambda)I_{2\lambda}^c\|_{p \rightarrow p} \leq C\|F\|_\infty$$

for all  $\lambda > 0$ .

*Proof.* By the definition of spectral multipliers

$$F(\mathcal{L}/\lambda)I_{2\lambda}^c f = \sum_{n=1}^{\infty} F(\lambda_n/\lambda) \phi_n \langle I_{2\lambda}^c f, \phi_n \rangle$$

so

$$\|F(\mathcal{L}/\lambda)I_{2\lambda}^c\|_{p \rightarrow p} \leq \sum_{n=1}^{\infty} |F(\lambda_n/\lambda)| \|\phi_n\|_p \|I_{2\lambda}^c \phi_n\|_{p'}.$$

Since  $\text{supp } F \subset [1/2, 1]$  in the sum above it is enough to consider only such  $n$  that  $\lambda_n \leq \lambda$ . It follows from Proposition 2.1 point (1) that  $1 < \lambda_n \leq \lambda$ . Hence by (2.6)

$$\|\phi_n\|_{L^p} \leq C\lambda^{1/2}$$

for all  $1 \leq p \leq \infty$ . Next by (2.5)

$$\|I_{2\lambda}^c \phi_n\|_{p'} \leq C \exp(-\lambda)$$

and

$$\|F(\mathcal{L}/\lambda)I_{2\lambda}^c\|_{p \rightarrow p} \leq C\lambda^{1/2} \exp(-\lambda) \left( \sum_{\lambda_n \leq \lambda} |F(\lambda_n/\lambda)| \right).$$

By (2.3) the number of eigenvalues below  $\lambda$  is of order  $\lambda^{3/2}$ , so

$$\sum_{\lambda_n \leq \lambda} |F(\lambda_n/\lambda)| \leq C\lambda^{3/2} \|F\|_\infty.$$

Thus

$$\|F(\mathcal{L}/\lambda)I_{2\lambda}^c\|_{p \rightarrow p} \leq C\lambda^2 \exp(-\lambda) \|F\|_\infty \leq C\|F\|_\infty.$$

This proves Lemma 3.1.  $\square$



Next we shall investigate operators  $F(\mathcal{L}/\lambda)I_{\lambda/4}$ , where as before  $F: \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function such that  $\text{supp } F \subset [1/2, 1]$ . We obtain sufficiently precise estimates for the norm  $\|F(\mathcal{L}/\lambda)I_{\lambda/4}\|_{p \rightarrow p}$ . However, the range of  $p$  for which the result holds is restricted to the interval  $1 \leq p \leq 2$  and the estimates involve the norm of function  $F$  in some Sobolev spaces  $H^s$  for  $s > 1/2$ . The proof of the following proposition follows closely an argument used in [10, Theorem 3.6] and [11, Theorem 3.2] which was partly motivated by results obtained by Mauceri, Meda, and Christ in [27, 6]. Some more developed versions of this idea are described in [5, Theorem 4.2] and in [32, Theorem 4.6].

**Proposition 3.2.** *Suppose that  $\mathcal{L}$  is an anharmonic oscillator defined by (1.1) and that  $F: \mathbb{R} \rightarrow \mathbb{R}$  is a function such that  $\text{supp } F \subset [1/2, 1]$  and  $F \in H^s(\mathbb{R})$  for some  $s > 1/2$ . Then for any  $1 \leq p \leq 2$*

$$\|F(\mathcal{L}/\lambda)I_{\lambda/4}\|_{p \rightarrow p} \leq C\|F\|_{H^s}$$

for all  $\lambda > 0$ .

**Remark 3.3.** Note that the same argument as in Section 6 below shows that Proposition 3.2 does not hold any longer if  $p > 4$ . More precisely for any  $p > 4$  there exists  $s > 1/2$  such that estimate from Proposition 3.2 is not satisfied.

*Proof of Remark 3.3.* Let  $\eta \in C_c^\infty(\mathbb{R})$  is such a function that  $\eta(0) = 1$  and  $\text{supp } \eta \subset [-\frac{\pi}{2}, \frac{\pi}{2}]$  and set  $F_n(\tau) = \eta((4\tau/3 - 1)\lambda_n \sqrt{\lambda_{n+1}})$ . Then for some small  $\epsilon$  and  $n$  large enough one has  $\text{supp } F_n \subset [3/4 - \epsilon, 3/4 + \epsilon]$ . Now if Proposition 3.2 holds for some  $p > 4$  then

$$\left\| F_n \left( \frac{\mathcal{L}}{\frac{4}{3}\lambda_n} \right) I_{\frac{4}{3}\lambda_n/4} \right\|_{p \rightarrow p} \leq C\|F_n\|_{H^s} \leq C\lambda_n^{3s/2-3/4}$$

However the same argument as in the proof of Theorem 6.1 shows that

$$\left\| F_n \left( \frac{\mathcal{L}}{\frac{4}{3}\lambda_n} \right) I_{\frac{4}{3}\lambda_n/4} \right\|_{p \rightarrow p} = \|I_{\lambda_n/3}\phi_n\|_{p'} \|\phi_n\|_p \geq c\lambda_n^{-1/4} \lambda_n^{1/p'-1/2}$$

Thus  $3s/2 - 3/4 > 1/p' - 3/4$  that is  $\frac{2}{3p'} < s$  and Proposition 3.2 cannot hold for any  $p > 4$ .  $\square$

The rest of this section is devoted to the proof of Proposition 3.2. We split its proof into a few separate statements. We start with recalling a useful notation coming from [10]. For any function  $F: \mathbb{R} \rightarrow \mathbb{R}$  and any parameter  $M \in (1, \infty)$  we set

$$(3.11) \quad \|F\|_{M,q} = \left( \frac{1}{M} \sum_{l=-\infty}^{\infty} \sup_{\theta \in [\frac{l-1}{M}, \frac{l}{M})} |F(\theta)|^q \right)^{1/q}.$$

The following lemma plays significant role in this section and in Section 5 below, see Lemma 5.3. Its proof is straightforward modification of the argument used in [10, (3.29)] and [11, Proposition 4.6].

**Lemma 3.4.** *Suppose that  $s > 0$  and that  $\xi \in C_c^\infty$  is a function such that  $\text{supp } \xi \subset [-1, 1]$ ,  $\widehat{\xi}(0) = 1$  and  $\widehat{\xi}^{(k)}(0) = 0$  for all  $1 \leq k \leq s+2$ . Next set  $\xi_M(\theta) = M\xi(M\theta)$  and assume that  $G: \mathbb{R} \rightarrow \mathbb{R}$ . Then*

$$\|G - G * \xi_M\|_{M,q} \leq CM^{-s} \|G\|_{W_s^q}.$$

for all  $s > 1/q$ . Moreover

$$\|G * \xi_M\|_{M,q} \leq \|G\|_q$$

and

$$\|G\|_{M,q}^q \leq C \left( \|G\|_q^q + M^{-qs} \|G\|_{W_s^q}^q \right).$$

*Proof.* For the proof of the first inequality we refer readers to [10, (3.29)] or [11, Proposition 4.6]. To show the second estimate note that

$$|\xi_M * G(\theta)|^q \leq \|\xi_M\|_{L^{q'}}^q \int_{\theta-1/M}^{\theta+1/M} |G(\theta')|^q d\theta',$$

so

$$\begin{aligned} \|\xi_M * G\|_{M,q} &= \left( \frac{1}{M} \sum_{i=-\infty}^{\infty} \sup_{\theta \in [\frac{i-1}{M}, \frac{i}{M})} |\xi_M * G(\theta)|^q \right)^{1/q} \\ &\leq \frac{\|\xi_M\|_{L^{q'}}}{M^{1/q}} \left( \sum_{i=-\infty}^{\infty} \int_{(i-2)/M}^{(i+1)/M} |G(\theta')|^q d\theta' \right)^{1/q} \leq \frac{3\|\xi_M\|_{L^{q'}}}{M^{1/q}} \|G\|_{L^q} \leq C \|G\|_{L^q}, \end{aligned}$$

see also [11, (4.9)]. This proves the second estimate. The third estimate is a direct consequence of first two.  $\square$

The next step in the proof of Proposition 3.2 is to establish some partial restriction type estimate result. Note that the global version (without projection  $I_{\lambda/4}$ ) of such restriction estimate is false. Indeed examining the proof of Proposition 4.8 below shows that without projection  $I_{\lambda/4}$  the Lemma 3.5 can only hold if the norm  $\|F\|_{\lambda^{3/2},2}^2$  is replaced by the stronger norm  $\|F\|_{\lambda^{3/2},4+\epsilon}^2$ . We want to point out also that one has to apply following estimates to the operator  $F * \xi_M$  so it is necessary to assume that we consider functions with support slightly outside the interval  $[1/2, 1]$ . Note also that  $\|F(\mathcal{L}/\lambda)\|_{p \rightarrow p} \leq C_{\lambda_0} \|F\|_{\infty}$  for all  $\lambda \leq \lambda_0$ , any fixed  $\lambda_0$  and all  $1 \leq p \leq \infty$ . In fact any  $L^p \rightarrow L^q$  norm satisfies such estimates. Hence it is enough to consider large  $\lambda$  that is  $\lambda$  bigger than some fixed constant.

**Lemma 3.5.** *Suppose that  $\mathcal{L}$  is an anharmonic oscillator defined by (1.1) and that  $\lambda > 4$ . Assume also that  $F: \mathbb{R} \rightarrow \mathbb{R}$  is a function such that  $\text{supp } F \subset [3/8, 9/8]$ .*

*Then for any  $1 \leq p \leq \infty$*

$$(3.12) \quad \|F(\mathcal{L}/\lambda)I_{\lambda/4}\|_{1 \rightarrow 2}^2 = \sup_{|y| \leq \lambda/4} \int_{\mathbb{R}} |K_{F(\mathcal{L}/\lambda)}(x, y)|^2 dx \leq C \lambda^{1/2} \|F\|_{\lambda^{3/2},2}^2$$

where  $\|F\|_{\lambda^{3/2},2}^2$  is the norm defined by (3.11) with  $M = \lambda^{3/2}$ .

*Proof.* By orthonormality of the eigenfunction expansion

$$\int_{\mathbb{R}} |K_{F(\mathcal{L}/\lambda)}(x, y)|^2 dx = \sum_{k=[3\lambda^{3/2}/8]+1}^{[9\lambda^{3/2}/8]+1} \|K_{(\chi_{[\frac{k-1}{\lambda^{3/2}}, \frac{k}{\lambda^{3/2}})})^{F(\mathcal{L}/\lambda)}(\cdot, y)}\|_{L^2}^2.$$

Note next that if  $\lambda_{n+1} \leq 9\lambda/8$  then by (2.4)

$$\lambda_{n+1} - \lambda_n \geq \frac{\pi}{2} \lambda_{n+1}^{-1/2} \geq \lambda^{-1/2}.$$

Hence there is at most one number of the form  $\lambda_n/\lambda$  which belongs to interval  $[\frac{k-1}{\lambda^{3/2}}, \frac{k}{\lambda^{3/2}})$ . Thus

$$\begin{aligned}
 \int_{\mathbb{R}} |K_{F(\mathcal{L}/\lambda)}(x, y)|^2 dx &= \sum_{k=[3\lambda^{3/2}/8]+1}^{[9\lambda^{3/2}/8]+1} \|K_{(\chi_{[\frac{k-1}{\lambda^{3/2}}, \frac{k}{\lambda^{3/2}})})^{F(\mathcal{L}/\lambda)}(\cdot, y)}\|_{L^2}^2 \\
 &= \sum_{k=[3\lambda^{3/2}/8]+1}^{[9\lambda^{3/2}/8]+1} \int_{\mathbb{R}} \left| \sum_{\lambda_n \in [\frac{k-1}{\lambda^{1/2}}, \frac{k}{\lambda^{1/2}})} F(\lambda_n/\lambda) \phi_n(x) \phi_n(y) \right|^2 dx \\
 (3.13) \quad &\leq \sum_{k=[3\lambda^{3/2}/8]+1}^{[9\lambda^{3/2}/8]+1} \sup_{\theta \in [\frac{k-1}{\lambda^{3/2}}, \frac{k}{\lambda^{3/2}})} |F(\theta)|^2 |\phi_n(y)|^2.
 \end{aligned}$$

The eigenfunction  $\phi_n$  in the last line of the estimates above corresponds to the unique  $\lambda_n$  such that  $\lambda_n \in [\frac{k-1}{\lambda^{1/2}}, \frac{k}{\lambda^{1/2}})$  and if such eigenvalue does not exist it should be replaced by 0. However if  $|y| \leq \lambda/4$  and  $\lambda_n \in [3\lambda/8, 9\lambda/8]$  then by (2.5)

$$|\phi_n(y)|^2 \leq C|\lambda_n|^{-1} \leq C\lambda^{-1}.$$

Thus (3.12) follows from (3.13). □

The next ingredient required for our main argument is a simple lemma described in [11]. Recall that for any positive potential  $V \in L^1_{loc}(\mathbb{R}^d)$  we can define the operator  $L = -\Delta_d + V$  by the standard quadratic forms approach.

**Lemma 3.6.** *Let  $L = -\Delta_d + V$ , where  $V \in L^1_{loc}(\mathbb{R}^d)$  and  $V \geq 0$ . Suppose that for some  $c > 0$*

$$\int_{\mathbb{R}^d} (1 + V(x))^{-c} dx < \infty.$$

*Then*

$$\|(1 + L)^{-c/2}\|_{L^2 \rightarrow L^1} < C \int_{\mathbb{R}^d} (1 + V(x))^{-c} dx.$$

*Proof.* For the proof we refer readers to [11, Lemma 7.9]. □

Following corollary is a straightforward consequence of Lemmas 3.5 and 3.6

**Corollary 3.7.** *Suppose that  $\mathcal{L}$  is an anharmonic oscillator defined by (1.1). Assume also that  $F: \mathbb{R} \rightarrow \mathbb{R}$  is a function such that  $\text{supp } F \subset [1/4, 2]$ . Then for every  $\varepsilon > 0$  there exists a constant  $C_\varepsilon$  such that*

$$\|F(\mathcal{L}/\lambda)I_{\lambda/4}\|_{1 \rightarrow 1}^2 \leq C_\varepsilon \lambda^{(3/2+\varepsilon)} \|F\|_{\lambda^{3/2}, 2}^2.$$

*for all  $\lambda > 4$ .*

*Proof.* It is enough to note that if  $c = 1 + \varepsilon$  and  $G(\theta) = (1 + \lambda\theta)^{c/2} F(\theta)$  then

$$\begin{aligned}
 \|F(\mathcal{L}/\lambda)I_{\lambda/4}\|_{1 \rightarrow 1}^2 &\leq \|(1 + \mathcal{L})^{c/2} F(\mathcal{L}/\lambda)I_{\lambda/4}\|_{1 \rightarrow 2}^2 \|(1 + \mathcal{L})^{-c/2}\|_{L^2 \rightarrow L^1}^2 \\
 &\leq C_\varepsilon \lambda^{1/2} \|G\|_{\lambda^{3/2}, 2}^2 \leq C_\varepsilon \lambda^{(3/2+\varepsilon)} \|F\|_{\lambda^{3/2}, 2}^2.
 \end{aligned}$$

□

The rest of the proof of Proposition 3.2 is now a straightforward modification of argument used in [10, Lemma 3.4] or in [11, Section 4]. Therefore here we only sketch the proof to show the role and significance of Lemma 3.5.

*Proof of Proposition 3.2.* It is enough to show that for any  $\varepsilon > 0$

$$\|F(\mathcal{L}/\lambda)I_{\lambda/4}\|_{1 \rightarrow 1} \leq C\|F\|_{H^{1/2+\varepsilon}}.$$

To do this consider function  $\xi_\lambda$  defined in Lemma 3.4 and set

$$F(\mathcal{L}/\lambda) = (F - F * \xi_{\lambda^{3/2}})(\mathcal{L}/\lambda) + F * \xi_{\lambda^{3/2}}(\mathcal{L}/\lambda).$$

Recall that  $\|F(\mathcal{L}/\lambda)\|_{p \rightarrow p} \leq C\|F\|_\infty$  for any  $\lambda < \lambda_0$ , any fixed  $\lambda_0$  and  $1 \leq p \leq \infty$ . Hence it is enough to consider large  $\lambda$  and we can assume that  $\text{supp } F * \xi_{\lambda^{3/2}} \subset [3/8, 9/8]$ . Now by Corollary 3.7 and Lemma 3.4

$$\begin{aligned} \|(F - F * \xi_{\lambda^{3/2}})(\mathcal{L}/\lambda)I_{\lambda/4}\|_{1 \rightarrow 1}^2 &\leq C\lambda^{3/2+\varepsilon}\|(F - F * \xi_{\lambda^{3/2}})\|_{\lambda^{3/2},2}^2 \\ &\leq C\lambda^{3/2+\varepsilon}\lambda^{-2(3/2)(1/2+\varepsilon/3)}\|F\|_{H^{1/2+\varepsilon/3}}^2 \\ (3.14) \quad &\leq C\|F\|_{H^{1/2+\varepsilon/3}}^2. \end{aligned}$$

To estimate the term corresponding to  $F * \xi_{\lambda^{3/2}}$  we note that by Lemmas 3.5 and 3.4

$$\|F * \xi_{\lambda^{3/2}}(\mathcal{L}/\lambda)I_{\lambda/4}\|_{1 \rightarrow 2}^2 \leq C\lambda^{1/2}\|F * \xi_{\lambda^{3/2}}\|_{\lambda^{3/2},2}^2 \leq C\lambda^{1/2}\|F\|_2^2.$$

Equivalently the above inequality can be stated as

$$(3.15) \quad \sup_{|y| \leq \lambda/4} \int_{\mathbb{R}} |K_{F * \xi_{\lambda^{3/2}}(\mathcal{L}/\lambda)}(x, y)|^2 dx \leq C\lambda^{1/2}\|F\|_2^2.$$

However we recall that for any operator satisfying Gaussian-type heat kernel bounds the following basic estimate holds, see [11, (4.4) and (4.5)] or [18]

$$\begin{aligned} \sup_{|y| \leq \lambda/4} \int_{\mathbb{R}} |K_{F * \xi_{\lambda^{3/2}}(\mathcal{L}/\lambda)}(x, y)|^2 (1 + \lambda^{1/2}|x - y|)^s dx &\leq C|B(y, \lambda^{1/2})| \|F * \xi_{\lambda^{3/2}}\|_{H^{(s+1)/2+\varepsilon}}^2 \\ &= C\lambda^{1/2}\|F * \xi_{\lambda^{3/2}}\|_{H^{(s+1)/2+\varepsilon}}^2 \leq C\lambda^{1/2}\|F\|_{H^{(s+1)/2+\varepsilon}}^2. \end{aligned}$$

Now one can use Mauceri-Meda interpolation trick, see [11, Lemma 4.3]. That is, we can consider the above estimates with large  $s$  and interpolate with inequality (3.15) to show that

$$\sup_{|y| \leq \lambda/4} \int_{\mathbb{R}} |K_{F * \xi_{\lambda^{3/2}}(\mathcal{L}/\lambda)}(x, y)|^2 (1 + \lambda^{1/2}|x - y|)^{1+\varepsilon'} dx \leq C\lambda^{1/2}\|F\|_{H^{1/2+\varepsilon''}}^2.$$

for all  $\varepsilon' < \varepsilon''$ . Alternatively one can prove that the above estimates follows from Lemma 3.5 using the finite propagation speed for the wave equation technique, see [10, (3.10) and (3.28)]. The last estimate and the Cauchy-Schwarz inequality yield

$$\sup_{|y| \leq \lambda/4} \int_{\mathbb{R}} |K_{F * \xi_{\lambda^{3/2}}(\mathcal{L}/\lambda)}(x, y)| dx \leq C\|F\|_{H^{1/2+\varepsilon''}}^2.$$

Thus

$$\|F * \xi_{\lambda^{3/2}}(\mathcal{L}/\lambda)I_{\lambda/4}\|_{1 \rightarrow 1}^2 \leq C\|F\|_{H^{1/2+\varepsilon}}^2.$$

This finishes the proof of Proposition 3.2.  $\square$

One of most surprising points of our approach is the fact that the optimal spectral multiplier result cannot be obtained as a consequence of restriction type estimates and one has to develop new techniques to obtain the optimal Bochner-Riesz summability exponent. To illustrate this point we will sketch the proof of Proposition 3.8 below even though this result does not give the sharp Bochner-Riesz summability result described in Theorem 1.3. Inspecting the proof it is easy to note that estimate (3.16) fails for  $q < 4$  so this approach does not lead to the optimal result for Bochner-Riesz summability. Property (3.16) is a discrete version of restriction type estimates close in nature to Sogge's cluster type estimates, see for example [10, Theorem 2.2 and Corollary 2.3]. It is interesting to note here that in the setting of the standard Laplace operator of classical Fourier Transform the Bochner-Riesz and Restriction conjecture are very closely related, see [39].

**Proposition 3.8.** *Suppose that operator  $\mathcal{L}$  is defined by formula (1.1) and assume also that  $\text{supp } F \subset [-1, 1]$ . Then for all  $1 \leq p \leq \infty$  and any  $s > 1/2$ ,*

$$\sup_{t>0} \|F(t\mathcal{L})\|_{L^p \rightarrow L^p} \leq C \|F\|_{W_s^4},$$

where  $\|F\|_{W_s^4} = \|(1 - d_x^2)^{s/2} F\|_4$  is the norm in  $L^4$  Sobolev space of order  $s$ .

*Proof.* It is enough to prove Proposition 3.8 for  $p=1$ . The rest of the range follows then by self-adjointness and interpolation. Using the same approach as in Proposition 3.2, Lemma 3.5 and Corollary 3.7 it is not difficult to note that to prove Proposition 3.8 it is enough to show the following version of estimate (3.12)

$$(3.16) \quad \|F(\mathcal{L}/\lambda)\|_{1 \rightarrow 2}^2 = \sup_y \int_{\mathbb{R}} |K_{F(\mathcal{L}/\lambda)}(x, y)|^2 dx \leq C \lambda^{1/2} \|F\|_{\lambda^{3/2}, q}^2$$

for  $q = 4 + \varepsilon$ , for all  $\varepsilon > 0$  and for all functions  $F$  such that  $\text{supp } F \subset [-1, 1]$ .

We are going to prove estimate (3.16) only for  $y = \lambda$  as the proof for other  $y \in \mathbb{R}$  is similar or simpler. By (3.13), estimate (2.5) and Hölder's inequality

$$\begin{aligned} \int_{\mathbb{R}} |K_{F(\mathcal{L}/\lambda)}(x, y)|^2 dx &= \sum_{k=1}^{[\lambda^{3/2}]+1} \sup_{\theta \in [\frac{k-1}{\lambda^{3/2}}, \frac{k}{\lambda^{3/2}})} |F(\theta)|^2 |\phi_n(y)|^2 \\ &\leq C \sum_{k=1}^{[\lambda^{3/2}]+1} \sup_{\theta \in [\frac{k-1}{\lambda^{3/2}}, \frac{k}{\lambda^{3/2}})} |F(\theta)|^2 \lambda^{-1/2} (|\lambda - k\lambda^{-1/2}| + 1)^{-1/2} \\ &\leq C \lambda^{1/2} \|F\|_{\lambda^{3/2}, 2p}^2 \left( \lambda^{-3/2} \sum_{k=1}^{[\lambda^{3/2}]+1} \left( \frac{\lambda}{\lambda|1 - k\lambda^{-3/2}| + 1} \right)^{p'/2} \right)^{1/p'}. \end{aligned}$$

Recall that the eigenfunction  $\phi_n$  in the above estimates corresponds to the unique  $\lambda_n$  such that  $\lambda_n \in [\frac{k-1}{\lambda^{1/2}}, \frac{k}{\lambda^{1/2}})$  and if such eigenvalue does not exist it should be replaced by 0. Note that the last sum is uniformly bounded independently of  $\lambda$  if  $p' < 2$ . This shows (3.16) for any  $q = 2p > 4$  and finishes the proof of Proposition 3.8.  $\square$

## 4. MORE LIGHT ON AIRY FUNCTION

Consider the Airy operator which formally defined by the formula

$$(4.17) \quad \mathcal{A} = -\frac{d^2}{dx^2} + x.$$

The Airy function  $\text{Ai}$  which we recall in Section 2 is a bounded on  $\mathbb{R}$  solution of the equation  $\mathcal{A}f = 0$ . Another linearly independent solution of this equation function  $\text{Bi}$  grows exponentially as  $x \rightarrow \infty$  so it is not a tempered distribution and is not relevant to our discussion here.

Using just function  $\text{Ai}$  we can describe complete system of eigenfunctions of  $\mathcal{A}$ . Set  $\varphi_\lambda(x) = \text{Ai}(x - \lambda)$ . For any function  $f \in L^2(\mathbb{R})$  we define the Airy Transform by the following formula

$$\mathcal{T}f(\lambda) = \langle f, \varphi_\lambda \rangle = (f * \check{\text{Ai}})(\lambda),$$

where  $\check{\text{Ai}}(\lambda) = \text{Ai}(-\lambda)$ .

Since  $\widehat{\text{Ai}}(\omega) = \exp(i\omega^3/3)$ , mapping  $\mathcal{T}$  is an isometry on  $L^2(\mathbb{R})$  and its inverse is given by

$$\mathcal{T}^{-1}g(x) = \text{Ai} * g(x).$$

for any  $g \in L^2(\mathbb{R})$ .

**Lemma 4.1.** *Suppose that  $F: \mathbb{R} \rightarrow \mathbb{R}$  is a bounded function and let  $F(\mathcal{A})$  be the spectral multiplier corresponding to function  $F$  and the Airy operator  $\mathcal{A}$ . Then*

$$\mathcal{T}(F(\mathcal{A})f)(\lambda) = F(\lambda)\mathcal{T}f(\lambda)$$

for all  $f \in L^2(\mathbb{R})$ . In addition  $K_{F(\mathcal{A})}(x, y) = F(\mathcal{A})\delta_y(x)$  - the kernel of the operator  $F(\mathcal{A})$  is described by the formula

$$K_{F(\mathcal{A})}(x, y) = \int_{-\infty}^{\infty} F(\lambda)\varphi_\lambda(x)\varphi_\lambda(y)d\lambda = \int_{-\infty}^{\infty} F(\lambda)\text{Ai}(x - \lambda)\text{Ai}(y - \lambda)d\lambda.$$

Moreover

$$\int_{\mathbb{R}} |K_{F(\mathcal{A})}(x, y)|^2 dy = \int_{\mathbb{R}} |F(\lambda)\text{Ai}(x - \lambda)|^2 d\lambda$$

for all  $x \in \mathbb{R}$ .

*Proof.* Lemma 4.1 follows from the definition of the Airy transform  $\mathcal{T}$  and the following simple observation

$$\mathcal{A}\varphi_\lambda = \lambda\varphi_\lambda$$

by a standard argument. □

In the sequel it will be convenient to use the following description of the asymptotic of the Airy function which for  $x$  negative is a slightly more precise version of estimate (2.8)

**Lemma 4.2.** *The Airy function can be expanded as*

$$(4.18) \quad \text{Ai}(x) = \exp(i\zeta(x))\theta(x) + \exp(-i\zeta(x))\bar{\theta}(x)$$

where  $\zeta(x) = 2|x|^{3/2}/3$  for  $x < -1$ . Moreover, function  $\zeta$  and all of its derivatives are bounded for  $x \geq -1$  and  $|d_x^k \theta|(x) \leq C_k(1 + |x|)^{-k-1/4}$  for all  $x \in \mathbb{R}$ .

*Proof.* Function  $\text{Ai}(x)$  is an entire analytic function of  $x \in \mathbb{C}$  and by [20, (7.6.18), Page 214] for all  $x \in \mathbb{C}$

$$\text{Ai}(x) = -\omega_1 \text{Ai}(\omega_1 x) - \omega_2 \text{Ai}(\omega_2 x)$$

where  $\omega_1 = e^{i\pi/3} = \frac{-1}{2} + \frac{\sqrt{3}}{2}i$  and  $\omega_2 = \omega_1^2 = \frac{-1}{2} - \frac{\sqrt{3}}{2}i$ . For  $x < -1$ , by [20, (7.6.19), Page 214],

$$\text{Ai}(\omega_1 x) = \exp(-2i|x|^{3/2}/3)(2\pi)^{-1} \int_{-\infty}^{\infty} \exp(-\xi^2 \sqrt{|x|} \omega_3 + i\xi^3/3) d\xi$$

and

$$\text{Ai}(\omega_2 x) = \exp(2i|x|^{3/2}/3)(2\pi)^{-1} \int_{-\infty}^{\infty} \exp(-\xi^2 \sqrt{|x|} \omega_4 + i\xi^3/3) d\xi$$

where  $\omega_3 = -\overline{\omega_1}$  and  $\omega_4 = \overline{\omega_2}$ . Thus for  $x < -1$ , (4.18) holds with

$$\theta(x) = -\omega_2 (2\pi)^{-1} \int_{-\infty}^{\infty} \exp(-\xi^2 \sqrt{|x|} \omega_4 + i\xi^3/3) d\xi.$$

For  $x \geq -1$ , set  $\zeta(x) = 1/(1+x^2)$  and  $\theta(x) = e^{-i/(1+x^2)} \text{Ai}(x)/2$ . By [20, (7.6.20), Page 215] function  $\text{Ai}$  and all its derivatives decay exponentially when  $x$  tends to  $+\infty$ . This finishes the proof of Lemma 4.2.  $\square$

The next statement is a standard oscillatory integral type estimates.

**Theorem 4.3.** *Suppose that  $\psi \in C^\infty(\mathbb{R})$  is a real valued function and that  $u \in C_c^\infty(V)$ , where  $V$  is a closed subset of  $\mathbb{R}$ . Then for each positive integer  $l > 0$  there exists a positive constant  $C_l$  such that for  $\lambda > 0$*

$$\left| \int \exp(i\lambda\psi(t)) u(t) dt \right| \leq C_l \sum_{k=0}^l \sup_V |u^{(k)}| |\psi'|^{k-2l} \lambda^{-l}.$$

The constant  $C_l$  in the above estimate is bounded when  $\psi$  stays in a bounded set in  $C^{l+1}(V)$ .

*Proof.* Theorem 4.3 is a special one-dimensional case of [20, Theorem 7.7.1] under additional assumption that function  $\psi$  is real valued.  $\square$

In the next statement we will describe estimates for the kernel  $K_{w(\mathcal{A})}(x, y) = w(\mathcal{A})\delta_y(x)$  of the spectral multiplier  $w(\mathcal{A})$  which play crucial role in the proof of our main result.

**Proposition 4.4.** *Let  $w \in C_c^\infty(\mathbb{R})$  be a smooth function such that  $\text{supp } w \subset [-a, a]$ . For all  $k \in \mathbb{N}$  choose constants  $C_k > 0$  in such a way that*

$$(4.19) \quad |d_x^k w|(x) \leq C_k a^{-k}$$

and  $C_k$  do not depend on  $a$ .

Then

A) For all  $x \in \mathbb{R}$  and  $y$  satisfying  $a \geq \min(1, |y|^{-1/2})$ ,

$$(4.20) \quad |K_{w(\mathcal{A})}(x, y)| = |w(\mathcal{A})\delta_y|(x) \leq C'_l \frac{d^{-1}}{(1 + |x - y|/d)^l} \left( 1 + \frac{|y|}{1 + |x|} \right)^{\frac{1}{4}}$$

where  $d = \max(a^{-1/2}, |y|^{1/2}/a)$  and  $C'_l$  just depends on the constants  $C_k$  in (4.19) and  $l$ , but not on  $a, x$  and  $y$ .

B) For all  $x \in \mathbb{R}$  and  $y$  satisfying  $a \leq \min(1, |y|^{-1/2})$ ,

$$(4.21) \quad |K_{w(\mathcal{A})}(x, y)| = |w(\mathcal{A})\delta_y|(x) \leq C_l'' \frac{a}{(1 + a^2|x|)^l} (1 + |y|)^{-1/4} (1 + |x|)^{-1/4}$$

where  $C_l''$  just depends on  $C_k$  in (4.19) and  $l$ , but not on  $a, x$  and  $y$ .

We shall prove Parts A) and B) separately.

*Proof of Part A).* Recall that in Part A) of Proposition 4.4 we assume that  $a \geq \min(1, |y|^{-1/2})$ . We split the proof of Part A) of Proposition 4.4 into two cases:

- I)  $|y| \leq a^4$ ;
- II)  $|y| > a^4$ .

**Case I:**  $|y| \leq a^4$ . In fact our argument in **Case I** yields a stronger version of inequality (4.20) mainly

$$(4.22) \quad |w(\mathcal{A})\delta_y|(x) \leq C_l' \frac{d^{-1}}{(1 + |x - y|/d)^l}.$$

Note that if  $|y| \leq a^4$  then immediately, combining the assumption  $a \geq \min(1, |y|^{-1/2})$ , we have  $a \geq 1$ . Put

$$h(\lambda) = \mathcal{T}(w(\mathcal{A})\delta_y)(\lambda) = w(\lambda)\text{Ai}(y - \lambda).$$

Next we calculate the Fourier transform of  $h$ ,

$$\begin{aligned} \hat{h}(\omega) &= \int \hat{w}(t) \exp(-i((\omega - t)^3/3 + y(\omega - t))) dt \\ &= \exp(-i(\omega^3/3 + y\omega)) \int \hat{w}(t) \exp(-i(-t\omega^2 + t^2\omega - t^3/3 - yt)) dt \\ &= \exp(-i(\omega^3/3 + y\omega)) g(\omega), \end{aligned}$$

where the last equality defines function  $g$ . Note that

$$\widehat{w(\mathcal{A})\delta_y}(\omega) = \widehat{\text{Ai} * h}(\omega) = \hat{\text{Ai}}(\omega) \hat{h}(\omega) = e^{-iy\omega} g(\omega).$$

Hence to prove estimate (4.22) it is enough to show that

$$(4.23) \quad |\hat{g}(x)| \leq C_l' d^{-1} (1 + |x|/d)^{-l}.$$

Recall that  $d = \max(|y|^{1/2}/a, a^{-1/2})$ . We make the following claim.

*Claim.* If  $g$  is a function defined above then there exist constants  $C_k'$  such that

$$(4.24) \quad |g^{(k)}|(\omega) \leq C_k' \frac{d^k}{1 + |\omega^2 + y|/a}$$

for all  $k \in \mathbb{N}$  and  $\omega \in \mathbb{R}$ .

First we observe that estimate (4.23) and in fact whole **Case I** follows from (4.24). Indeed set  $\omega_y = \sqrt{\max(0, -y)}$  and note that

$$\min(|\omega - \omega_y|^2, |\omega + \omega_y|^2) \leq |\omega - \omega_y| |\omega + \omega_y| \leq |\omega^2 + y|$$

Hence

$$\int_{\mathbb{R}} \frac{1}{1 + |\omega^2 + y|/a} d\omega \leq \int_{\mathbb{R}} \left( \frac{1}{1 + |\omega - \omega_y|^2/a} + \frac{1}{1 + |\omega + \omega_y|^2/a} \right) d\omega = Ca^{1/2}.$$



It follows now from estimates (4.24) and the relation  $d = \max(|y|^{1/2}/a, a^{-1/2}) \geq a^{-1/2}$  that

$$\|g^{(k)}\|_1 \leq CC'_k d^k a^{1/2} \leq CC'_k d^{k-1}.$$

Now the estimates (4.23) and **Case I** are straightforward consequence of  $L^1$  estimates of derivatives of  $g$  stated above. Hence to finish the proof of **Case I** it is enough to show estimate (4.24).

*Proof of Claim (4.24).* Set

$$\psi(\omega, t) = a^{-1}(\omega^2 + y)t - a^{-2}\omega t^2 + a^{-3}t^3/3.$$

Substituting  $t/a$  for  $t$  we have

$$g(\omega) = \int u(t) \exp(i\psi(\omega, t)) dt$$

where  $u(t) = \hat{w}(t/a)/a$ . Now let  $\eta$  be a smooth cutoff function such that  $\text{supp}(\eta) \subset [-1, 1]$  and  $\sum_{j \in \mathbb{Z}} \eta(t-j) = 1$  for all  $t \in \mathbb{R}$ . Write  $u(t) = \sum_{j \in \mathbb{Z}} u_j(t) = \sum_{j \in \mathbb{Z}} u(t)\eta(t-j)$ . Decompose  $g$  as

$$g(\omega) = \sum_{j \in \mathbb{Z}} g_j(\omega) = \sum_{j \in \mathbb{Z}} \int u_j(t) \exp(i\psi(\omega, t)) dt.$$

Now to prove (4.24) for  $k = 0$  it is clearly enough to show that for some natural  $N_1 \geq 2$  and every  $N_2 \geq 1$

$$(4.25) \quad |g_j|(\omega) \leq \frac{C_{N_1, N_2} (1 + |j|)^{-N_1}}{(1 + |\omega^2 + y|/a)^{N_2}}.$$

with constant  $C_{N_1, N_2}$  independent of  $j$ . In fact in the case  $k = 0$  it is enough to consider term  $1 + |\omega^2 + y|/a$  instead of  $(1 + |\omega^2 + y|/a)^{N_2}$ , but we will have to verify a bit stronger estimate when we consider the case  $k > 0$ , see (4.27) below. Next, assumption (4.19) on function  $w$  and the fact that  $\text{supp } u_j \subset [j-1, j+1]$  yields

$$|d_t^k u_j(t)| \leq C_{k, N} (1 + |j|)^{-N}$$

for all  $k \in \mathbb{N}$ . Hence

$$|g_j(\omega)| \leq \int_{j-1}^{j+1} |u_j(t)| dt \leq C_N (1 + |j|)^{-N}.$$

Now if  $|\omega^2 + y|/a \leq 32(1 + |j|)^2$  then (4.25) is a straightforward consequence of the above estimate so we can assume further on that  $|\omega^2 + y|/a > 32(1 + |j|)^2$ .

If this is the case we want to estimate  $g_j(\omega)$  as an oscillatory integral. When  $|\omega^2 + y|/a > 32(1 + |j|)^2$ , the following inequalities hold for all  $t \in [j-1, j+1]$ ,

$$(4.26) \quad \left| \frac{\omega t}{a(\omega^2 + y)} \right| < 1/4 \quad \text{and} \quad \left| \frac{t^2}{a^2(\omega^2 + y)} \right| < 1/4.$$

Indeed, since  $a \geq 1$  we have

$$\left| \frac{t^2}{a^2(\omega^2 + y)} \right| = \frac{a^{-3}t^2}{|(\omega^2 + y)/a|} \leq \frac{(1 + |j|)^2}{|(\omega^2 + y)/a|} < 1/4.$$

When  $|\omega^2 + y| \geq \omega^2/2$  then

$$\left| \frac{\omega t}{a(\omega^2 + y)} \right|^2 \leq \frac{|\omega|^2}{|\omega^2 + y|} \frac{(1 + |j|)^2}{|(\omega^2 + y)/a|} < 1/16.$$

When  $|\omega^2 + y| < \omega^2/2$ , then  $|y| > \omega^2/2$ ,  $|\omega| \leq 2|y|^{1/2}$  and

$$\left| \frac{\omega t}{a(\omega^2 + y)} \right| \leq \frac{2|y|^{1/2}(1 + |j|)}{a^2(|\omega^2 + y|/a)} < 1/4$$

where we used inequality  $|y|^{1/2}/a^2 \leq 1$ . These calculations verify (4.26).

Write  $\psi(\omega, t) = a^{-1}(\omega^2 + y)\psi_1(\omega, t)$  where

$$\psi_1(\omega, t) = t - \frac{\omega}{a(\omega^2 + y)}t^2 + \frac{t^3}{3a^2(\omega^2 + y)}.$$

We have

$$\partial_t \psi_1(\omega, t) = 1 - 2\frac{\omega t}{a(\omega^2 + y)} + \frac{t^2}{a^2(\omega^2 + y)}.$$

Thus by (4.26)  $\partial_t \psi_1(\omega, t) > 1/4$  and all higher derivatives of  $\psi_1$  are bounded. Substituting  $\psi = \psi_1$ ,  $u = u_j$  and  $\lambda = a^{-1}(\omega^2 + y)$  in Theorem 4.3 yields estimate (4.25). This proves (4.24) for  $k = 0$ .

To handle  $k > 0$  note that  $\partial_\omega \psi(\omega, t) = 2a^{-1}\omega t - a^{-2}t^2$  and

$$\partial_\omega^k \exp(i\psi(\omega, t)) = P_k(\omega, t) \exp(i\psi(\omega, t))$$

where  $P_k(\omega, t)$  is a polynomial such that

$$|\partial_t^l P_k(\omega, t)| \leq C_{k,l}(|\omega|^k a^{-k} + a^{-2k} + a^{-k/2})(1 + |t|)^{2k}$$

for all  $t \in \mathbb{R}$ . In fact,  $P_1 = i(2a^{-1}\omega t - a^{-2}t^2)$ ,  $P_{k+1} = P_1 P_k + \partial_\omega P_k$  and one can inductively prove that  $P_k = \sum_{l,j \in N_k} b_{k,l,j} \omega^l (t/a)^j$  where  $N_k$  is set of points with integer coordinates in the triangle with vertices  $(k, k)$ ,  $(0, 2k)$ ,  $(0, k/2)$ . To see that  $(i, j)$  is above or on line trough  $(k, k)$  and  $(0, k/2)$  assign to  $\omega^l t^j$  degree  $-l/2 + j$  and note that minimal degree of term in  $P_{k+1}$  is bigger by  $1/2$  then minimal degree of term in  $P_k$ . Considering normal degree  $l + j$  we see that  $(l, j)$  is below line trough  $(k, k)$  and  $(0, 2k)$  which shows that indeed  $P_k$  is of prescribed form. Now, we estimate each term of  $P_k$  separately using inequality between arithmetic and geometric mean.

Next

$$\begin{aligned} d_\omega^k g_j(\omega) &= \int_{j-1}^{j+1} u_j(t) \partial_\omega^k \exp(i\psi(\omega, t)) dt \\ &= \int_{j-1}^{j+1} P_k(\omega, t) u_j(t) \exp(i\psi(\omega, t)) dt. \end{aligned}$$

Repeating the argument which we use above to prove (4.25) with  $u_j$  replaced by  $P_k(\omega, t)u_j$  yields

$$|d_\omega^k g_j(\omega)| \leq C'_k(|\omega|^k a^{-k} + a^{-2k} + a^{-k/2}) \frac{(1 + |j|)^{-2}}{(1 + |\omega^2 + y|/a)^{N_2}}.$$

Since  $a^{-1/2} \leq d$  and  $a \geq 1$  so  $a^{-2k} \leq a^{-k/2} \leq d^k$ . Hence if

$$(4.27) \quad (|\omega|/a)(1 + |\omega^2 + y|/a)^{-1/2} \leq Cd$$

then claim (4.24) is satisfied for all  $k$ .

To see that (4.27) holds we consider two cases. When  $|\omega^2 + y| \geq \omega^2/2$ , then

$$|\omega|a^{-1}(1 + |\omega^2 + y|/a)^{-1/2} \leq |\omega|a^{-1}(\omega^2/(2a))^{-1/2} = 2^{1/2}a^{-1/2}.$$

When  $|\omega^2 + y| < \omega^2/2$ , then  $|\omega| \leq 2|y|^{1/2}$  and

$$|\omega|a^{-1} \leq 2|y|^{1/2}/a \leq 2d$$

so indeed (4.27) holds. This ends the proof of estimates (4.24) and **Case I**.

**Case II:**  $|y| > a^4$ . In this case,  $|y| \geq 1$ ,  $|y| \geq a^2$  so  $d = |y|^{1/2}/a \geq 1$ . Recall that by Lemma 4.1

$$(w(\mathcal{A})\delta_y)(x) = \int w(s)\text{Ai}(y-s)\text{Ai}(x-s)ds.$$

We further split **Case II** into four sub-cases.

*Case (i):*  $y < -a - 1$  and  $x < -a - 1$  then by equality (4.18)

$$\begin{aligned} (w(\mathcal{A})\delta_y)(x) &= \int \exp \left[ i(\zeta(y-s) + \zeta(x-s)) \right] w_1(s)\theta(x-s)ds \\ &\quad + \int \exp \left[ i(\zeta(y-s) - \zeta(x-s)) \right] w_1(s)\bar{\theta}(x-s)ds \\ &\quad + \int \exp \left[ -i(\zeta(y-s) + \zeta(x-s)) \right] w_2(s)\bar{\theta}(x-s)ds \\ &\quad + \int \exp \left[ -i(\zeta(y-s) - \zeta(x-s)) \right] w_2(s)\theta(x-s)ds \\ &= I_1 + I_2 + I_3 + I_4, \end{aligned}$$

where  $w_1(s) = w(s)\theta(y-s)$  and  $w_2(s) = w(s)\bar{\theta}(y-s)$ .

We will only estimate  $I_2$  because the proofs for the other integrals are similar. We write

$$\begin{aligned} I_2 &= \int \exp \left[ i(\zeta(y-s) - \zeta(x-s)) \right] w_1(s)\bar{\theta}(x-s)ds \\ &= a \int \exp \left[ i(\zeta(y-at) - \zeta(x-at)) \right] u(t,x)dt \end{aligned}$$

where  $u(t,x) = w(at)\theta(y-at)\bar{\theta}(x-at)$ . We estimate  $I_2$  again using the approach of oscillatory integrals.

Next we consider three ranges of variable  $x, y \in \mathbb{R}$ :  $|x| > 2|y|$ ,  $|y|/2 \leq |x| \leq 2|y|$  and  $|x| \leq |y|/2$ . If  $|x| > 2|y|$  then by Lemma 4.2 and assumption (4.19),

$$|\partial_t^k u(t,x)| \leq C'_k(1+|x-at|)^{-1/4}(1+|y-at|)^{-1/4} \leq C'_k|y|^{-1/2}$$

where we use the facts that,  $\text{supp}(u(\cdot, x)) \subset [-1, 1]$  so we can assume that  $|t| \leq 1$ ; and that if  $a \leq 2$  then  $|y| > 1 + a \geq 3a/2$ , otherwise if  $a > 2$  then  $|y| > a^4 > 2a$ . Now if  $|x-y|a|y|^{-1/2} \leq 1$  then

$$I_2 \leq Ca|y|^{-1/2} \leq Ca|y|^{-1/2}C'_l(1+|x-y|a|y|^{-1/2})^{-l} \leq C'_l d^{-1}(1+|x-y|/d)^{-l}.$$

Thus we can assume  $|x-y|a|y|^{-1/2} > 1$ . In addition in the consider range of  $x, y$  we have  $\zeta(z) = 2(-z)^{3/2}/3$  so

$$\partial_t(\zeta(y-at) - \zeta(x-at)) = a((-y+at)^{1/2} - (-x+at)^{1/2}).$$

Then absolute value of the derivative is bounded below by  $c(|x-y|a|y|^{-1/2})^{1/2}$  since  $|y| \geq a$ ,  $a \geq |y|^{-1/2}$  and  $|x| \sim |x-y|$ . Also note that by  $|y| \geq a^4$ , if  $|x-y|a|y|^{-1/2} > 1$ ,  $|x-y| > |y|^{1/2}/a \geq a$ . Next we observe that higher derivatives of the phase function are uniformly

bounded by constant  $C(|x - y|a|y|^{-1/2})^{1/2}$ . The lower bounds for derivative verified above shows that we can choose  $\lambda \sim (|x - y|a|y|^{-1/2})^{1/2}$  in a such way that  $\partial_t(\zeta(y - at) - \zeta(x - at))/\lambda \geq 1$ . Substituting  $\psi = (\zeta(y - at) - \zeta(x - at))/\lambda$  in Theorem 4.3 yields

$$I_2 \leq a|y|^{-1/2}C'_l(1 + |x - y|a|y|^{-1/2})^{-l} \leq C'_l d^{-1}(1 + |x - y|/d)^{-l}.$$

For the range  $|y|/2 \leq |x| \leq 2|y|$ , we use the similar argument as above. For the function  $u$ , we still have

$$|\partial_t^k u(t, x)| \leq C'_k(1 + |x - at|)^{-1/4}(1 + |y - at|)^{-1/4} \leq C'_k|y|^{-1/2}.$$

However we have to modify required estimates for the phase function. As before, we can assume  $|x - y|a|y|^{-1/2} > 1$  and in the consider range of  $x, y$  we have

$$\partial_t(\zeta(y - at) - \zeta(x - at)) = a((-y + at)^{1/2} - (-x + at)^{1/2}).$$

Then absolute value of the derivative is bounded below by  $c|x - y|a|y|^{-1/2}$  since  $|y| \geq a$ ,  $a \geq |y|^{-1/2}$  and  $|x| \sim |y|$ . Next we observe that higher derivatives of the phase function are uniformly bounded by constant  $C|x - y|a|y|^{-1/2}$  since  $|y| \geq a$ ,  $|x| \sim |y|$  and  $|x - y|a|y|^{-1/2} > 1$ . The lower bounds for derivative verified above shows that we can choose  $\lambda \sim |x - y|a|y|^{-1/2}$  in a such way that  $\partial_t(\zeta(y - at) - \zeta(x - at))/\lambda \geq 1$ . Substituting  $\psi = (\zeta(y - at) - \zeta(x - at))/\lambda$  in Theorem 4.3 yields

$$I_2 \leq a|y|^{-1/2}C'_l(1 + |x - y|a|y|^{-1/2})^{-l} \leq C'_l d^{-1}(1 + |x - y|/d)^{-l}.$$

For the range  $|x| \leq |y|/2$  we use the same argument as before with the same phase function  $\psi = (\zeta(y - at) - \zeta(x - at))/\lambda$ . However we have to modify required estimates for  $\lambda$  and the function  $u$ .

If  $|x| \leq |y|/2$ ,  $a \geq 2$  and  $|x| < 2a$ , then note that  $a \geq 2 > \min\{1, |x|^{-1/2}\}$ ,  $|x| < 2a \leq a^4$  and  $(w(\mathcal{A})\delta_y)(x) = (w(\mathcal{A})\delta_x)(y)$ . By the result of **Case I**, we have that for  $d' = \max\{a^{-1/2}, |x|^{1/2}/a\} \leq 1$ ,

$$(w(\mathcal{A})\delta_y)(x) = (w(\mathcal{A})\delta_x)(y) \leq C'_l d'^{-1}(1 + |x - y|/d')^{-l-1}.$$

Note that  $d \geq 1$ ,  $d' \leq 1$ ,  $a \geq 2$ ,  $|y| \geq 1$  and  $|x - y| \sim |y|$ , then

$$(w(\mathcal{A})\delta_y)(x) \leq C'_l a|x - y|^{-1}(1 + |x - y|/d')^{-l} \leq C'_l a|y|^{-1/2}(1 + |x - y|/d)^{-l}.$$

If  $|x| \leq |y|/2$ ,  $a \geq 2$  and  $|x| \geq 2a$ , we get

$$|\partial_t^k u(t, x)| \leq C'_k|y|^{-1/4}.$$

and  $\text{supp}(u(\cdot, x)) \subset [-1, 1]$ . Note that  $|x| \leq |y|/2$  implies that  $|x - y| \sim |y|$ . Thus the absolute value of the derivative of the phase function is bounded below by  $ca|y|^{1/2}$  and the higher derivative of the phase function is bounded by  $Ca|y|^{1/2}$ . Then as before, choosing  $\lambda \sim a|y|^{1/2}$ , by Theorem 4.3

$$\begin{aligned} I_2 &\leq a|y|^{-1/4}C'_l(a|y|^{1/2})^{-l-1} \\ &\leq C'_l a|y|^{-1/2}(a|y|^{1/2})^{-l} a^{-1}|y|^{-1/4} \\ &\leq C'_l a|y|^{-1/2}(1 + a|y|^{1/2})^{-l} \\ &\leq C'_l a|y|^{-1/2}(1 + a|x - y||y|^{-1/2})^{-l} \\ &\leq C'_l d^{-1}(1 + |x - y|/d)^{-l}. \end{aligned}$$

If  $|x| \leq |y|/2$  and  $a < 2$ , we get  $|x| \geq 3a/2$  and thus

$$|\partial_t^k u(t, x)| \leq C'_k |y|^{-1/4} (1 + |x|)^{-1/4}$$

and  $\text{supp}(u(\cdot, x)) \subset [-1, 1]$ . Note that  $|x| \leq |y|/2$  implies that  $|x - y| \sim |y|$ . Thus the absolute value of the derivative of the phase function is bounded below by  $ca|y|^{1/2}$  and the higher derivative of the phase function is bounded by  $Ca|y|^{1/2}$ . Then as before, choosing  $\lambda \sim a|y|^{1/2}$ , by Theorem 4.3

$$\begin{aligned} I_2 &\leq a|y|^{-1/4} (1 + |x|)^{-1/4} C'_l (a|y|^{1/2})^{-l} \leq C'_l a |y|^{-1/2} (1 + a|x - y||y|^{-1/2})^{-l} \frac{|y|^{1/4}}{(1 + |x|)^{1/4}} \\ &\leq C'_l d^{-1} (1 + |x - y|/d)^{-l} \left( \frac{|y|}{1 + |x|} \right)^{\frac{1}{4}}. \end{aligned}$$

*Case (ii):*  $y \geq -a - 1$  and  $x < -a - 1$ . For  $|s| \leq a$ , when  $a < 2$ ,  $y - s \geq -a - 1 - a \geq -5$  and so all derivatives of  $\text{Ai}(y - s)$  are exponential decay. Thus for all  $k, l, l' \in \mathbb{N}$

$$\begin{aligned} |\partial_s^k (w(s) \text{Ai}(y - s))| &\leq \sum_{m \leq k} C_{m,l} a^{-m} (1 + a^{-1}|s|)^{-l} (1 + |y - s|)^{-l'} \\ (4.28) \quad &\leq C'_{k,l} a^{-k} (1 + a^{-1}|s|)^{-l} (1 + |y|)^{-l'}. \end{aligned}$$

When  $a \geq 2$ , by  $|y| > a^4$ ,  $y$  must be bigger than 0 and  $y - s > a^4 - a > 2a > 0$ . Thus all derivatives of  $\text{Ai}(y - s)$  are exponential decay. Then for all  $k, l, l' \in \mathbb{N}$

$$\begin{aligned} |\partial_s^k (w(s) \text{Ai}(y - s))| &\leq \sum_{m \leq k} C_{m,l} a^{-m} (1 + a^{-1}|s|)^{-l} (1 + |y - s|)^{-(l' + k - m)} \\ &\leq \sum_{m \leq k} C_{m,l} a^{-m} (1 + |y - s|)^{m - k} (1 + a^{-1}|s|)^{-l} (1 + |y - s|)^{-l'} \\ (4.29) \quad &\leq C'_{k,l} a^{-k} (1 + a^{-1}|s|)^{-l} (1 + |y|)^{-l'}. \end{aligned}$$

When  $|y| > |x|/2$ , by estimates (4.28), (4.29) and  $d \geq 1$

$$\begin{aligned} |(w(\mathcal{A})\delta_y)(x)| &= \left| \int w(s) \text{Ai}(y - s) \text{Ai}(x - s) ds \right| \\ &\leq 2aC (1 + |y|)^{-l'} \\ &\leq Ca|y|^{-1/2} (1 + |x - y|)^{-l} \\ &\leq Cd^{-1} (1 + |x - y|/d)^{-l}. \end{aligned}$$

When  $|y| \leq |x|/2$  by (4.18)

$$\begin{aligned} (4.30) \quad (w(\mathcal{A})\delta_y)(x) &= \int w(s) \text{Ai}(y - s) \text{Ai}(x - s) ds \\ &= \int h(s) \theta(x - s) e^{i(2((-x+s)^{3/2})/3)} ds \\ &\quad + \int h(s) \bar{\theta}(x - s) e^{-i(2((-x+s)^{3/2})/3)} ds \\ &= a \int h(at) \theta(x - at) e^{i(2((-x+at)^{3/2})/3)} dt \end{aligned}$$

$$+a \int h(at)\bar{\theta}(x-at)e^{-i(2((-x+at)^{3/2})/3)}dt.$$

For function  $h(at)\theta(x-at)$  or  $h(at)\bar{\theta}(x-at)$ , by estimates (4.28), (4.29) and Lemma 4.2, all the derivatives are bounded by  $C'_k|y|^{-1/2}$ . For the phase function  $2((-x+at)^{3/2})/3$ , since  $|y| \geq a$  and  $|x| \geq 2|y| \geq 2a$ ,  $\partial_t(2((-x+at)^{3/2})/3) = a((-x+at)^{1/2})$  is bounded below by  $ca|x|^{1/2}$  and all higher derivatives are bounded by  $Ca|x|^{1/2}$ . Then by Theorem 4.3 and  $a|x|^{1/2} \geq a^{1/2}|y|^{-1/4}|x-y|^{1/2}$ ,

$$\begin{aligned} (w(\mathcal{A})\delta_y)(x) &\leq aC'_l|y|^{-1/2}(a|x|^{1/2})^{-l} \\ &\leq C'_la|y|^{-1/2}(1+a|y|^{-1/2}|x-y|)^{-l/2} \\ &\leq C'_ld^{-1}(1+|x-y|/d)^{-l/2}. \end{aligned}$$

For *Case (iii)*:  $y \geq -a-1, x \geq -a-1$ , when  $|y| > |x|/2$ , the proof is similar to that in the situation  $|y| > |x|/2$  of *Case (ii)*; when  $|y| \leq |x|/2$ , because  $|x| \geq 2|y| > 2a^4$ , the estimate (4.20) follows from that both  $\text{Ai}(x-s)$  and  $\text{Ai}(y-s)$  decay exponentially.

For *Case (iv)*:  $y < -a-1, x \geq -a-1$ , when  $|x| > |y|/2$ , the proof is similar to that in the situation  $|y| > |x|/2$  of *Case (ii)*; when  $|x| \leq |y|/2$ ,  $a > 2$  and  $|x| \leq 2a$ , the proof is similar to that in the situation  $|x| \leq |y|/2$ ,  $a \geq 2$  and  $|x| \leq 2a$  of *Case (i)*; for the other situations, the proof is similar to that in the situation  $|y| \leq |x|/2$  of *Case (ii)*.  $\square$

Next we discuss the proof of Part B of Proposition 4.4.

*Proof of Part B*). Recall that in Part B) of Proposition 4.4 we assume that  $a \leq \min(1, |y|^{-1/2})$ . It is not difficult to notice that for  $x \geq -2$  estimate (4.21) is straightforward consequence of exponential decay of the Airy function for positive argument. Hence we only consider  $x \leq -2$ .

Note that if  $|s| \leq a$  then  $|\text{Ai}(y-s)| \leq C(1+|y|)^{-1/4}$  and otherwise  $w(s) = 0$ . Next, in the considered case  $a \leq 1$  and  $|y|^{1/2} \leq a^{-1}$  so it follows from Lemma 4.2 that

$$\begin{aligned} |\partial_s \text{Ai}(y-s)| &\leq C(1+|y-s|^{1/2})(1+|y-s|)^{-1/4} \leq C \max\{1, |y|^{1/2}\}(1+|y|)^{-1/4} \\ &\leq Ca^{-1}(1+|y|)^{-1/4} \end{aligned}$$

for all  $|s| \leq a$ . Inductively, using the defining relation  $\text{Ai}''(x) = x\text{Ai}(x)$ , we get

$$|\partial_s^k \text{Ai}(y-s)| \leq Ca^{-k}(1+|y|)^{-1/4} \quad \forall |s| \leq a.$$

Now it follows from assumptions on  $w$  (that is  $\text{supp } w \subset [-a, a]$  and (4.19)) that the function  $h(s) = w(s)\text{Ai}(y-s)$  satisfies the estimate

$$|\partial_s^k(h(s))| \leq C'_{k,l}a^{-k}(1+a^{-1}|s|)^{-l}(1+|y|)^{-1/4}.$$

Using the above inequality, then writing

$$\begin{aligned} w(\mathcal{A})\delta_y(x) &= \int w(s)\text{Ai}(y-s)\text{Ai}(x-s)ds = \int w(s)\text{Ai}(y-s)\theta(x-s)e^{i(2(-x+s)^{3/2}/3)}ds \\ &\quad + \int w(s)\text{Ai}(y-s)\bar{\theta}(x-s)e^{-i(2(-x+s)^{3/2}/3)}ds \end{aligned}$$

and setting  $u(t) = w(s)\text{Ai}(y-s)\theta(x-s)$  or  $u(t) = w(s)\text{Ai}(y-s)\bar{\theta}(x-s)$  yield estimate (4.21) by oscillatory integrals argument of Theorem 4.3.  $\square$

## 5. PROOF OF THEOREM 1.2

This section is entirely devoted to the proof of Theorem 1.2. Because the argument is rather complex we divide it into several steps formulated as separate statements. First in Lemmas 5.1 and 5.2 we split the multiplier  $F$  into large but analytic part  $G$  and small but rough part  $H$ . Next in Lemma 5.3 we estimate  $L^1 \rightarrow L^1$  norm of  $H(\mathcal{L}/\lambda)$  and in Lemma 5.4 we discuss  $L^{4/3} \rightarrow L^{4/3}$  norm of the same operator. To obtain these  $L^{4/3}$  estimates we “interpolate” between  $L^1$  and  $L^2$  but the interpolation argument is not standard. Later we use a similar interpolation trick in the proof of  $L^{4/3}$  estimates for  $G(\mathcal{L}/\lambda)$  in Lemma 5.13. In Lemma 5.5 we describe further wavelet like decomposition of the nice (analytic) part  $G$ . This decomposition allow us to apply estimates for the Airy operator which we obtained in Section 4 to study the multiplier  $G(\mathcal{L}/\lambda)$ , see Lemma 5.9 below. In fact a bit earlier in Lemma 5.6, based on the finite speed propagation property for the wave equation, we show that in crucial part of our argument we can replace multiplier  $G(\mathcal{L}/\lambda)$  by the multipliers corresponding to the Airy operator.

Our aim is to investigate bounds for  $\|F(\mathcal{L}/\lambda)\|_{p \rightarrow p}$  for  $1 \leq p \leq 2$  and for large  $\lambda$ . For  $p \geq 2$ , we use duality and for small  $\lambda$  any required estimates hold, see the discussion before Lemma 3.5. We want to estimate the kernel  $F(\mathcal{L}/\lambda)\delta_y$ , where  $F \in H^s$ ,  $s > 1/2$  and  $\text{supp } F \subset [1/2, 1]$ . By Proposition 3.2 we know that  $\|F(\mathcal{L}/\lambda)I_{\lambda/4}\|_{p \rightarrow p} \leq C\|F\|_{H^{1/2+\epsilon}}$  for all  $1 \leq p \leq 2$  so we can assume that  $|y| > \lambda/4$ . It follows also from the obvious symmetry of the considered operator  $\mathcal{L}$  that we can also assume that  $y > \lambda/4 > 0$  without loss of generality. In addition it follows from Lemma 3.1 that in the proof we only need to estimate the norm of the restricted operator  $\|I_{2\lambda}F(\mathcal{L}/\lambda)I_{2\lambda}\|_{p \rightarrow p}$  for  $1 \leq p \leq 2$ .

We write

$$F(\mathcal{L}/\lambda) = \tilde{F}(\sqrt{\mathcal{L}/\lambda}) = \tilde{F}(\lambda^{-1/2}\sqrt{\mathcal{L}})$$

where  $\tilde{F}(x) = F(x^2)$ . Let  $\psi$  be a function such that  $\hat{\psi}$  is smooth,  $\text{supp } \hat{\psi} \subset [-1, 1]$ ,  $\hat{\psi} = 1$  on  $[-1/2, 1/2]$ ,  $0 \leq \hat{\psi} \leq 1$ , and  $\hat{\psi}$  is symmetric. Next for  $h > 0$  we set  $\psi_h(x) = h\psi(hx)$  and we define function  $\tilde{G}$  in the following way

$$\tilde{G} = \tilde{F} * \psi_{\lambda^{3/2}/6}.$$

Note that

$$\|\tilde{G}\|_{H^s} \leq C\|\tilde{F}\|_{H^s} \leq C\|F\|_{H^s}$$

and

$$\|\tilde{F} - \tilde{G}\|_{H^s} \leq C\|\tilde{F}\|_{H^s} \leq C\|F\|_{H^s}$$

Now we define functions  $F$  and  $G$  by the following formula

$$(5.31) \quad G(x) = \tilde{G}(\sqrt{x}) \quad \text{and} \quad H = F - G.$$

Note that functions  $G$  and  $H$  depend on choice of  $\lambda$ . In the rest of this section  $\lambda$  is treated as fixed large constant. In Lemmas 5.1 and 5.2 below we derive some straightforward differentiability properties of  $G$  and  $H$  which we use to estimate “tail” parts of spectral multipliers of  $G(\mathcal{L}/\lambda)$  and  $H(\mathcal{L}/\lambda)$ .

**Lemma 5.1.** *Assume that  $\lambda > 1$ ,  $\text{supp } F \subset [1/2, 1]$ ,  $F \in L^2(\mathbb{R})$  and let  $G$  be the function corresponding to  $F$  and  $\lambda$  defined by (5.31). Then  $G$  can be extended to an entire analytic function and there exists a constant  $C > 0$  such that*

$$|G(z)| \leq C\lambda^{3/2} \exp(\lambda^{3/2}|z|^{1/2}/6) \|F\|_{L^2}$$

for all  $z \in \mathbb{C}$ .

*Proof.* Note that since  $\text{supp}(\hat{\psi}) \subset [-1, 1]$  and  $\int |\hat{\psi}| \leq 1$ ,  $\psi$  is an entire analytic function satisfying

$$|\psi(z)| \leq \exp(|\text{Im}(z)|).$$

Consequently  $\tilde{G}$  is an entire function and

$$|\tilde{G}|(z) \leq C\lambda^{3/2} \exp(\lambda^{3/2} |\text{Im}(z)|/6) \|F\|_{L^2}$$

(the last inequality follows since the  $L^2$  norms of  $F$  and  $\tilde{F}$  are comparable). Note that both  $\tilde{F}$  and  $\psi$  are symmetric so  $\tilde{G}$  is also symmetric. Hence  $G$  is a well defined entire function. Thus

$$\begin{aligned} |G(z)| = |\tilde{G}(\sqrt{z})| &\leq C\lambda^{3/2} \exp(\lambda^{3/2} |\text{Im}(\sqrt{z})|/6) \|F\|_{L^2} \\ &\leq C\lambda^{3/2} \exp(\lambda^{3/2} |z|^{1/2}/6) \|F\|_{L^2}. \end{aligned}$$

This ends the proof of Lemma 5.1.  $\square$

In the next lemma we describe the behavior and of  $L^2$  norm of function  $H$  depending on  $\lambda$  and we notice that outside the support of  $F$  function  $H$  decays rapidly.

**Lemma 5.2.** *Assume that  $\lambda > 1$ ,  $\text{supp } F \subset [1/2, 1]$ ,  $s \geq 0$ ,  $F \in H^s(\mathbb{R})$  and let  $G$  and  $H$  be the functions corresponding to  $F$  and  $\lambda$  defined by (5.31). Then there exists a constant  $C$  such that*

$$\|H\|_{L^2(\mathbb{R}_+)} \leq C(\lambda^{3/2})^{-s} \|F\|_{H^s}.$$

In addition there exists a constant  $C$  such that,

$$\sup_{0 \leq x \leq 1/4} (|H(x^2)| + |d_x(H(x^2))| + |d_x^2(H(x^2))|) \leq C\|F\|_{L^2}$$

and for every  $l \in \mathbb{N}$  there exists a constant  $C_l$  such that

$$\sup_{x > 2} (|H(x)| + |xd_x H(x)| + |x^2 d_x^2 H(x)|) x^l \leq C_l \lambda^{-l} \|F\|_{L^2}$$

for all  $\lambda > 1$ .

*Proof.* We have  $\|\tilde{F}\|_{H^s} \leq C\|F\|_{H^s}$ , so  $\|\tilde{G}\|_{H^s} \leq C\|F\|_{H^s}$ . Since Fourier transform of  $\psi_{\lambda^{3/2}/6}$  is 1 on  $[-\lambda^{3/2}/12, \lambda^{3/2}/12]$ , we have

$$\|\tilde{F} - \tilde{G}\|_{L^2} \leq (\lambda^{3/2}/12)^{-s} \|\tilde{F}\|_{H^s} \leq C(\lambda^{3/2})^{-s} \|F\|_{H^s}.$$

Changing variables yields

$$\begin{aligned} (5.32) \quad \int_0^4 |H|^2(x) dx &= \int_0^4 |\tilde{F} - \tilde{G}|^2(\sqrt{x}) dx \leq 4\|\tilde{F} - \tilde{G}\|_{L^2}^2 \\ &\leq C(\lambda^{3/2})^{-2s} \|F\|_{H^s}^2. \end{aligned}$$

Since  $\psi$  belongs to set of Schwartz class functions and  $\text{supp } \tilde{F} \subset [-1, -1/\sqrt{2}] \cup [1/\sqrt{2}, 1]$  so for every  $l \in \mathbb{N}$  there exists constant  $C_l$  such that

$$|H(x^2)| = |\tilde{F} - \tilde{G}|(x) \leq C_l(\lambda^{3/2}x)^{-l} \|F\|_{L^2}$$

for all  $x > 2$ . Thus for  $l'$  large enough

$$\int_4^\infty |H(x)|^2 dx \leq C\lambda^{-l'} \|F\|_{L^2}^2 \leq (\lambda^{3/2})^{-2s} \|F\|_{L^2}^2$$



which together with (5.32) gives the first estimate for  $L^2$  norm of  $H$ .

To show the second estimate we note that  $\text{supp } \tilde{F}$  and the interval  $[0, 1/4]$  are disjoint so for any  $l \in \mathbb{N}$  there exists a constant  $C_l$  such that for all  $x \leq 1/4$

$$|d_x^l(H(x^2))| = |d_x^l(\tilde{F} - \tilde{G})|(x) = |d_x^l \tilde{G}|(x) \leq C \|F\|_{L^2}$$

for all  $\lambda \geq 1$ .

To show the third estimate we note that for any  $l', l'' \in \mathbb{N}$  there exists constant  $C = C_{l', l''}$  such that for all  $x \geq 2^{1/2}$

$$|d_x^{l'}(\tilde{F} - \tilde{G})|(x) = |d_x^{l'} \tilde{G}|(x) \leq C(\lambda^{3/2}x)^{-l''} \|F\|_{L^2}.$$

Now the third estimate follows from the above inequality.  $\square$

**Lemma 5.3.** *Let  $\mathcal{L}$  be an anharmonic oscillator defined by (1.1) and assume next that  $s \geq 1/2 + 1/6$ ,  $\text{supp } F \subset [1/2, 1]$ ,  $F \in H^s(\mathbb{R})$ . Now if  $H$  is the function corresponding to  $F$  and  $\lambda$  defined by (5.31) then for all  $\lambda > 1$*

$$\|H(\mathcal{L}/\lambda)\delta_y\|_{L^1(I_{2\lambda})} \leq C \|F\|_{H^s}$$

where  $I_{2\lambda} = [-2\lambda, 2\lambda]$ .

*Proof.* Consider function  $\nu \in C_c^\infty(\mathbb{R})$  such that  $\nu(x) = 1$  for all  $x \in [1/8, 2]$  and  $\text{supp } \nu \subset [1/16, 4]$ . The second claim and the third claim of Lemma 5.2 show that for  $x \geq 2$  and  $x < 1/8$  function  $(1 - \nu)H$  satisfies assumptions of most standard multiplier theorems and the corresponding spectral multiplier  $(1 - \nu)H(\mathcal{L}/\lambda)$  satisfies estimate of Lemma 5.3, see e.g [11, Theorem 3.2]. [10, Theorem 3.2] or [17, Theorem 2.4]. To be more precise we choose a function  $\omega \in C_c^\infty(\mathbb{R})$  such that  $0 \leq \omega(x) \leq 1$  and  $\omega(x) = 1$  for  $x \leq 2$  and  $\omega(x) = 0$  for  $x \geq 4$ . For  $n \in \mathbb{N}$  put  $\omega_n(x) = \omega(2^{-n}x) - \omega(2^{-n+1}x)$  so that

$$\omega(x) + \sum_{n=1}^{\infty} \omega_n(x) = 1 \quad \forall x \geq 1.$$

Now if we set

$$H_{0,\nu} = \omega(1 - \nu)H \quad \text{and} \quad H_{n,\nu} = \omega_n(1 - \nu)H$$

then by Lemma 5.2

$$\|\delta_{2^{n+2}} H_{n,\nu}\|_{W_s^\infty} \leq C 2^{-n} \|F\|_2$$

for all  $s > 1/2$  and all  $n \in \mathbb{N}$ . Recall that  $W_s^\infty$  is  $L^\infty$  Sobolev space of order  $s$ . Note also that  $\text{supp } \delta_{2^{n+2}} H_{n,\nu} \subset [-1, 1]$ . Now by [5, Theorem 3.1] (or by the results from [17, 10, 11] mentioned above)

$$\begin{aligned} \|(1 - \nu)H(\mathcal{L}/\lambda)\|_{1 \rightarrow 1} &\leq \sum_{n=0}^{\infty} \|H_{n,\nu}(\mathcal{L}/\lambda)\|_{1 \rightarrow 1} \\ &= \sum_{n=0}^{\infty} \|\delta_{2^{n+2}} H_{n,\nu}(2^{-n-2}\mathcal{L}/\lambda)\|_{1 \rightarrow 1} \leq C \sum_{n=0}^{\infty} 2^{-n} \|F\|_2. \end{aligned}$$

Hence the operator  $(1 - \nu)H(\mathcal{L}/\lambda)$  is continuous on all  $L^p(\mathbb{R})$  spaces and it is enough to consider the the multiplier  $\nu H(\mathcal{L}/\lambda)$ . Note that  $\text{supp } \nu H \subset [1/16, 4]$ . Recall that  $\lambda_n$  and

$\phi_n$  are the eigenvalues and corresponding eigenfunctions of the operator  $\mathcal{L}$ . Next write

$$\begin{aligned} \|\nu H(\mathcal{L}/\lambda)\delta_y\|_{L^2}^2 &= \sum_{n=1}^{\infty} |\nu H(\lambda_n/\lambda)|^2 |\phi_n(y)|^2 \\ &\leq C\lambda^{-1/2} \sum_{n=1}^{\infty} |\nu H(\lambda_n/\lambda)|^2. \end{aligned}$$

By inequalities (2.3) and (2.4) of Proposition 2.1 the distance between  $\lambda_n$  and  $\lambda_{n+1}$  is of order  $\lambda_n^{-1/2} \sim \lambda^{-1/2}$ . Hence by Lemmas 3.4 and 5.2

$$\begin{aligned} \sum_{n=1}^{\infty} |\nu H(\lambda_n/\lambda)|^2 &\leq C \sum_{k=0}^{\infty} \sup\{|\nu H(x)|^2 : k\lambda^{-3/2} \leq x \leq (k+1)\lambda^{-3/2}\} \\ (5.33) \quad &\leq C'(\lambda^{3/2}\|\nu H\|_{L^2}^2 + (\lambda^{3/2})^{1-2s}\|\nu H\|_{H^s}^2) \\ &\leq C''(\lambda^{3/2})^{1-2s}\|F\|_{H^s}^2 \end{aligned}$$

Now,  $s \geq 1/2 + 1/6$ , so  $1 - 2s \leq -2/6$  and  $(\lambda^{3/2})^{1-2s} \leq (\lambda^{3/2})^{-2/6} = \lambda^{-1/2}$  which means that

$$\sum_{n=1}^{\infty} |\nu H(\lambda_n/\lambda)|^2 \leq C\lambda^{-1/2}\|F\|_{H^s}^2$$

which in turn implies

$$\|\nu H(\mathcal{L}/\lambda)\delta_y\|_{L^2}^2 \leq C_1\lambda^{-1/2}C_2\lambda^{-1/2}\|F\|_{H^s}^2 = C\lambda^{-1}\|F\|_{H^s}^2.$$

Hence by Hölder inequality

$$\begin{aligned} \|\nu H(\lambda^{-1}\mathcal{L})\delta_y\|_{L^1(I_{2\lambda})} &\leq |I_{2\lambda}|^{1/2}\|\nu H(\lambda^{-1}\mathcal{L})\delta_y\|_{L^2} \\ &\leq (4\lambda)^{1/2}(C\lambda^{-1/2})\|F\|_{H^s} \leq C'\|F\|_{H^s} \end{aligned}$$

which yields the claim.  $\square$

**Lemma 5.4.** *Let  $\mathcal{L}$  be an anharmonic oscillator defined by (1.1) and assume that  $s > 1/2$ ,  $\text{supp } F \subset [1/2, 1]$ ,  $F \in H^s(\mathbb{R})$  and let  $H$  be the function corresponding to  $F$  and  $\lambda$  defined by (5.31). Then*

$$\|H(\mathcal{L}/\lambda)\|_{L^{4/3}(I_{2\lambda}) \rightarrow L^{4/3}(I_{2\lambda})} \leq C_s\|F\|_{H^s}$$

for all  $\lambda \geq 1$ .

*Proof.* Similarly as in Lemma 5.3 it is enough to consider the operator  $\nu H(\mathcal{L}/\lambda)$  where  $\text{supp } (\nu H) \subset [1/16, 4]$ . We recall that

$$\nu H(\mathcal{L}/\lambda)f = \sum_{\lambda/16 \leq \lambda_n \leq 4\lambda} \nu H(\lambda_n/\lambda)\phi_n\langle f, \phi_n \rangle$$

Let now  $\eta \in C_c^\infty(\mathbb{R})$  be a such function that  $0 \leq \eta \leq 1$ ,  $\text{supp } \eta \subset [-2, -1/2] \cup [1/2, 2]$  and

$$\sum_{j \in \mathbb{Z}} \eta(2^{-j}x) = 1$$

for all  $x \neq 0$ . For  $x \geq 0$  put

$$\phi_{j,\lambda_n}(x) = \eta(2^{-j}(x - \lambda_n))\phi_n(x) \quad \text{for } j > 0 \text{ and } \phi_{0,\lambda_n} = \phi_n - \sum_{j>0} \phi_{j,\lambda_n}.$$

For  $x < 0$  we put  $\phi_{j,\lambda_n}(x) = 0$ . Next for all  $j \geq 0$  set

$$Q_j f = \sum_{\lambda/16 \leq \lambda_n \leq 4\lambda} \nu H(\lambda_n/\lambda) \phi_n \langle f, \phi_{j,\lambda_n} \rangle.$$

If  $\text{supp}(f) \subset I_{2\lambda}$ , then

$$\nu H(\mathcal{L}/\lambda) f = \sum_{2^j \leq 10\lambda} (Q_j + Q'_j) f$$

where  $Q'_j$  is build like  $Q_j$  but using parts of  $\phi_n$  on  $(-\infty, 0)$ . Next, by estimate (2.5)

$$|\phi_{j,\lambda_n}|(y) \leq C\lambda^{-1/4} 2^{-j/4}.$$

Consequently if  $K_{Q_j}(x, y) = Q_j \delta_y(x)$  is the kernel of the operator  $Q_j$  then,

$$\begin{aligned} \|K_{Q_j}(\cdot, y)\|_2^2 &= \|Q_j \delta_y\|_{L^2}^2 = \sum_{\lambda/16 \leq \lambda_n \leq 4\lambda} |\nu H(\lambda_n/\lambda)|^2 |\phi_{j,\lambda_n}|^2(y) \|\phi_n\|_{L^2}^2 \\ &\leq C\lambda^{-1/2} 2^{-j/2} \sum_{\lambda/16 \leq \lambda_n \leq 4\lambda} |\nu H(\lambda_n/\lambda)|^2 \end{aligned}$$

so by (5.33)

$$\|Q_j \delta_y\|_{L^1(I_{2\lambda})} \leq C\lambda^{1/4} 2^{-j/4} (\lambda^{-3/2})^{s-1/2} \|F\|_{H^s}.$$

Thus

$$\|I_{2\lambda} Q_j\|_{L^1 \rightarrow L^1(I_{2\lambda})} \leq C\lambda^{1/4} 2^{-j/4} (\lambda^{-3/2})^{s-1/2} \|F\|_{H^s}.$$

Next we consider the  $L^2$  norm of the operator  $Q_j$ . Note that

$$\|\phi_{j,\lambda_n}\|_{L^2}^2 \leq \lambda^{-1/2} 2^{j/2}$$

so

$$\begin{aligned} \|Q_j f\|_{L^2}^2 &= \sum_{\lambda/16 \leq \lambda_n \leq 4\lambda} |\nu H(\lambda_n/\lambda)|^2 |\langle f, \phi_{j,\lambda_n} \rangle|^2 \\ &\leq \|f\|_{L^2}^2 \lambda^{-1/2} 2^{j/2} \sum_{\lambda/16 \leq \lambda_n \leq 4\lambda} |\nu H(\lambda_n/\lambda)|^2 \end{aligned}$$

and

$$\|Q_j\|_{L^2 \rightarrow L^2} \leq C\lambda^{-1/4} 2^{j/4} (\lambda^{-3/2})^{s-1/2} \|F\|_{H^s}$$

Now, by interpolation

$$\|Q_j\|_{L^{4/3}(I_{2\lambda}) \rightarrow L^{4/3}(I_{2\lambda})} \leq C(\lambda^{-3/2})^{s-1/2} \|F\|_{H^s}$$

Hence

$$\sum_{2^j \leq 10\lambda} \|Q_j\|_{L^{4/3}(I_{2\lambda}) \rightarrow L^{4/3}(I_{2\lambda})} \leq C \log(\lambda + 1) (\lambda^{-3/2})^{s-1/2} \|F\|_{H^s}$$

which is bounded when  $s > 1/2$ . We get estimate for  $Q'_j$  by symmetry which ends the proof.  $\square$

We now move to estimates for the part of the multiplier corresponding to the function  $G$ . For any  $k \in \mathbb{Z}$ ,  $k \geq 0$  we define a set  $\Delta_k \subset \mathbb{Z}$  by the formula

$$(5.34) \quad \Delta_k = \{m \in \mathbb{Z} : 0 \leq m \leq 2^{k+2}\}.$$

In the next lemma we describe useful wavelet like decomposition of the function  $G$ .

**Lemma 5.5.** *Let  $\lambda > 1$  and  $G$  be the function corresponding to  $F$  and  $\lambda$  defined by (5.31). Assume also that  $s > 1/2$ . Then one can decompose function  $G$  in the following way*

$$G(x) = G_{-\infty}(x) + G_{\infty}(x) + \sum_{0 \leq k \leq \log_2(\lambda^{3/2}/6)} \sum_{m \in \Delta_k} G_{k,m}(x)$$

with functions  $G_{k,m}$  satisfying the following conditions:

$$(5.35) \quad \text{supp } G_{k,m} \subset [(m-1)2^{-k}, (m+1)2^{-k}],$$

$$(5.36) \quad |d_x^l G_{k,m}(x)| \leq C_l \Theta_{k,m} 2^{kl} \quad \forall l \in \mathbb{N},$$

and

$$(5.37) \quad \sum_{m \in \Delta_k} \Theta_{k,m}^2 \leq C 2^{-k(2s-1)} \|F\|_{H^s}^2,$$

where  $C_l$  are constants depending only on  $s$  and  $l$  but do not depend on  $k$ .

In addition  $\text{supp } (G_{-\infty}) \subset (-\infty, 1/16]$ ,  $|G_{-\infty}|(x) \leq |G(x)|$  and

$$\|x^3 G_{\infty}\|_{H^2} \leq C \|F\|_{H^s}.$$

*Proof.* We define  $G_{-\infty}$  and  $G_{\infty}$  multiplying  $G$  by a smooth cutoff function, in such a way that  $G_{-\infty} = G$  for  $x < 1/32$  and  $G_{\infty}(x) = G(x)$  for  $x > 3$ ,  $\text{supp } (G_{\infty}) \subset [2, \infty)$ . We assume that  $\text{supp } (F) \subset [1/2, 1]$  so  $G(x) = -H(x)$  for  $x > 1$ . Thus estimate for  $G_{\infty}$  is a consequence of the last claim of Lemma 5.2. Set  $J = G - G_{-\infty} - G_{\infty}$ . Then  $\text{supp } (J) \subset [1/32, 3]$ ,

$$\|J\|_{H^s} \leq C \|F\|_{H^s}$$

and

$$(5.38) \quad \int |d_x^l J|^2 \leq C_l ((\lambda)^{3/2})^{2(l-s)} \|F\|_{H^s}^2$$

for all  $l \geq s$ . Note that the last inequality for  $\tilde{G}$  follows by construction, since we cut off frequencies higher than  $(\lambda)^{3/2}/6$  from its Fourier transform. Changing variable yields the required estimates for  $J$ . Next let  $\eta$  be a smooth function which is 1 on  $\text{supp } J$  and such that  $\text{supp } \eta \subset [0, 7/2]$ . Recall that  $\psi$  is such a function that  $\hat{\psi}$  is smooth,  $\text{supp } (\hat{\psi}) \subset [-1, 1]$ ,  $\hat{\psi} = 1$  on  $[-1/2, 1/2]$ ,  $0 \leq \hat{\psi} \leq 1$ , and  $\hat{\psi}$  is symmetric and that  $\psi_h(x) = h\psi(hx)$

Set

$$J_0(x) = \eta(x)(J * \psi)(x).$$

Next we write

$$J_k(x) = \eta(x)(J * (\psi_{2^k} - \psi_{2^{k-1}}))(x)$$

for  $1 \leq k \leq \log_2(\lambda^{3/2}/6) - 1$  and

$$J_{k_0} = J - \sum_{0 \leq k \leq \log_2(\lambda^{3/2}/6) - 1} J_k$$

where  $k_0$  is an integer such that  $\lambda^{3/2}/12 < 2^{k_0} \leq \lambda^{3/2}/6$ . It follows from the definition of  $J_k$  that  $\text{supp } J_k \subset [0, 7/2]$ ,

$$J = \sum_{0 \leq k \leq \log_2(\lambda^{3/2}/6)} J_k$$

and

$$(5.39) \quad \int |d_x^l J_k|^2 \leq C_l 2^{2(l-s)k} \|F\|_{H^s}^2.$$

Note that to get the last inequality for  $k = k_0$  we use (5.38). Now let  $u$  be a smooth function such that  $u = 1$  on  $[0, 1/2]$ ,  $\text{supp}(u) \subset [-1/2, 1]$  and  $\sum_{m \in \mathbb{Z}} u(x - m) = 1$  for all  $x \in \mathbb{R}$ .

Set

$$G_{k,m}(x) = J_k(x)u(2^k x - m).$$

Since  $\text{supp}(J_k) \subset [0, 7/2]$  so  $G_{k,m} = 0$  for any  $m \notin \Delta_k$ . Next put

$$\Gamma_{k,l,m} = \sup_x |d_x^l G_{k,m}(x)|$$

and note that

$$2^{-k} \sum_{m \in \Delta_k} (\Gamma_{k,l,m})^2 \leq C \sum_{j=0}^l 2^{2(l-j)k} \|d_x^j J_k\|_{2^k, 2}^2$$

where  $\|d_x^j J_k\|_{2^k, 2}^2$  is the norm considered in Lemma 3.4. Now by Lemma 3.4

$$\|d_x^j J_k\|_{2^k, 2}^2 \leq C(\|d_x^j J_k\|_2^2 + 2^{-2k} \|d_x^j J_k\|_{H^1}^2) \leq C'(\|d_x^j J_k\|_2^2 + 2^{-2k} \|d_x^{j+1} J_k\|_2^2)$$

Hence (5.39) yields

$$\sum_{m \in \Delta_k} \Gamma_{k,l,m}^2 \leq C_l 2^{-k(2s-2l-1)} \|F\|_{H^s}^2.$$

Next set

$$\Theta_{k,m} = \sum_{l=0}^{\infty} 2^{-(k+2)l} C_l^{-1/2} \Gamma_{k,l,m}.$$

Then by the Cauchy-Schwarz inequality

$$\begin{aligned} \sum_{m \in \Delta_k} \Theta_{k,m}^2 &= \sum_{m \in \Delta_k} \left( \sum_{l=0}^{\infty} 2^{-(k+2)l} C_l^{-1/2} \Gamma_{k,l,m} \right)^2 \leq C \sum_{m \in \Delta_k} \left( \sum_{l=0}^{\infty} 2^{-2l} \right) \left( \sum_{l=0}^{\infty} C_l^{-1} 2^{-2(k+1)l} \Gamma_{k,l,m}^2 \right) \\ &\leq C \sum_{m \in \Delta_k} \sum_{l=0}^{\infty} C_l^{-1} 2^{-2(k+1)l} \Gamma_{k,l,m}^2 \leq C \sum_{l=0}^{\infty} 2^{-2(k+1)l} 2^{-k(2s-2l-1)} \|F\|_{H^s}^2 \\ &\leq C 2^{-k(2s-1)} \|F\|_{H^s}^2. \end{aligned}$$

Next we observe that

$$\begin{aligned} \sup_x |d_x^{l'} G_{k,m}(x)| &= c_{k,l',m} C_{l'}^{-1/2} 2^{-(k+2)l'} C_{l'}^{1/2} 2^{(k+2)l'} \\ &\leq \sum_{l=0}^{\infty} 2^{-(k+2)l} C_l^{-1/2} \Gamma_{k,l,m} C_{l'}^{1/2} 2^{(k+2)l'} \\ &\leq 4 C_{l'}^{1/2} \Theta_{k,m} 2^{kl'}. \end{aligned}$$

This proves the requested estimates for  $G_{k,m}$ .  $\square$

The next lemma is based on the finite propagation speed property for the wave equation.

**Lemma 5.6.** *Assume that  $0 < \lambda/4 \leq y$ ,  $\text{supp } F \subset [1/2, 1]$  and let  $G$  be the function corresponding to  $F$  and  $\lambda$  defined by (5.31). Then*

$$G(\mathcal{A}/\lambda)\delta_y = G(\mathcal{L}/\lambda)\delta_y$$

where  $\mathcal{A}$  is the Airy operator and  $\mathcal{L}$  is the anharmonic operator defined by (1.1)

*Proof.* It follows from the finite propagation speed property for the wave equation that

$$\cos t\sqrt{\mathcal{A}}\delta_y = \cos t\sqrt{\mathcal{L}}\delta_y$$

for all  $|t| \leq y$ . Next if  $F$  is an even function, then by the Fourier inversion formula,

$$F(\sqrt{\mathcal{L}}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{F}(t) \cos(t\sqrt{\mathcal{L}}) dt.$$

Recall now that  $G(z) = \tilde{G}(\sqrt{z})$  and that the support of the Fourier transform of  $\tilde{G}$  is contained in the interval  $[-\lambda^{3/2}/6, \lambda^{3/2}/6]$ , that is  $\text{supp } (\tilde{G})^\wedge \subset [-\lambda^{3/2}/6, \lambda^{3/2}/6]$ . Hence if we assume that  $0 < \lambda/4 \leq y$  then

$$\begin{aligned} G(\mathcal{L}/\lambda)\delta_y &= \tilde{G}(\sqrt{\mathcal{L}/\lambda})\delta_y = \frac{1}{2\pi} \int_{-\infty}^{\infty} (\tilde{G})^\wedge(t) \cos(t\sqrt{\mathcal{L}/\lambda})\delta_y dt \\ &= \frac{1}{2\pi} \int_{-\lambda^{3/2}/6}^{\lambda^{3/2}/6} (\tilde{G})^\wedge(t) \cos(t\sqrt{\mathcal{L}/\lambda})\delta_y dt = \frac{1}{2\pi} \int_{-\lambda^{3/2}/6}^{\lambda^{3/2}/6} (\tilde{G})^\wedge(t) \cos(t\sqrt{\mathcal{A}/\lambda})\delta_y dt \\ &= \tilde{G}(\sqrt{\mathcal{A}/\lambda})\delta_y = G(\mathcal{A}/\lambda)\delta_y. \end{aligned}$$

This ends the proof of Lemma 5.6.  $\square$

Now similarly as in Lemma 3.1 we set  $\tilde{I}_\lambda = [\lambda, \infty)$  and define  $\chi_{\tilde{I}_\lambda}$  as the characteristic function of the half-line  $\tilde{I}_\lambda$ . Then we denote by  $\tilde{I}_\lambda$  also a projection acting on  $L^p(\mathbb{R})$  spaces defined by

$$\tilde{I}_\lambda f(x) = \chi_{\tilde{I}_\lambda} f(x).$$

Using this notation we can state Lemma 5.6 in the following way

$$G(\mathcal{L}/\lambda)\tilde{I}_{\lambda/4} = G(\mathcal{A}/\lambda)\tilde{I}_{\lambda/4}.$$

Note that in virtue of Proposition 3.2 we can assume that  $y \in \tilde{I}_{\lambda/4}$  and this allows us to replace multipliers of the operator  $\mathcal{L}$  by spectral multipliers of the Airy operator  $\mathcal{A}$  by Lemma 5.6. Note next that by Lemma 3.1 it is enough to consider only a  $L^1(I_{2\lambda})$  portion of the whole  $L^1$  norm of the considered kernel.

**Lemma 5.7.** *Let  $\mathcal{A}$  be the Airy operator defined by (4.17) and  $G_{-\infty}$  be the function defined in Lemma 5.5. Then there exists a constant  $C > 0$ , such that*

$$\|G_{-\infty}(\mathcal{A}/\lambda)\delta_y\|_{L^1(I_{2\lambda})} \leq C\|F\|_{L^2}$$

for all  $\lambda \geq 4$  and all  $y \geq \lambda/4$ .

*Proof.* By Lemma 4.1

$$G_{-\infty}(\mathcal{A}/\lambda)\delta_y(\cdot) = \text{Ai} * (G_{-\infty}(\lambda^{-1}\cdot)\tilde{\text{Ai}}(\cdot - y)).$$

Recall that by (2.7) we have  $|\tilde{\text{Ai}}(x - y)| \leq C \exp(-(2/3)(|x| + |y|)^{3/2})$  for all  $x \leq 0$  and  $y \geq \lambda/4 > 0$ . Since  $|G_{-\infty}| \leq |G|$  so by Lemma 5.1

$$|G_{-\infty}(\lambda^{-1}x)\tilde{\text{Ai}}(x - y)| \leq C\lambda^{3/2} \exp(\lambda^{1/2}|x|^{1/2}/6) \exp(-(2/3)(|x| + |y|)^{3/2})\|F\|_{L^2}.$$

By the inequality between arithmetic and geometric means  $\lambda^{1/2}|x|^{1/2} \leq (\lambda + |x|)/2$ . Since  $|y| \geq \lambda/4$ , we get  $\lambda^{1/2}|x|^{1/2} \leq |y| + |x|$ . Thus if  $|y| > 1$  then

$$\lambda^{1/2}|x|^{1/2}/6 \leq (1/3)(|x| + |y|)^{3/2}$$

so

$$\begin{aligned} |G_{-\infty}(\lambda^{-1}x)\check{\text{Ai}}(x-y)| &\leq C\lambda^{3/2}\exp(-(1/3)(|x| + |y|)^{3/2})\|F\|_{L^2} \\ &\leq C\lambda^{3/2}\exp(-(1/6)|y|^{3/2})\exp(-(1/6)|x|^{3/2})\|F\|_{L^2}. \end{aligned}$$

Hence

$$\|G_{-\infty}(\lambda^{-1}\cdot)\check{\text{Ai}}(\cdot - y)\|_{L^2((-\infty, 0])} \leq C'\lambda^{3/2}\exp(-\lambda^{3/2}/24)\|F\|_{L^2}.$$

Next for  $x \in [0, 1/16]$  by the second claim of Lemma 5.2

$$|G(x)| = |-H(x)| \leq \sup_{0 \leq x \leq 1/4} |H(x^2)| \leq C'\|F\|_{L^2},$$

so  $|G_{-\infty}(x)| \leq C'\|F\|_{L^2}$ . Consequently for  $x \in [0, \lambda/16]$

$$\begin{aligned} |G_{-\infty}(\lambda^{-1}x)\check{\text{Ai}}(x-y)| &\leq C\exp(-(1/3)(|x-y|^{3/2})\|F\|_{L^2} \\ &\leq C\exp(-\lambda^{3/2}/24)\|F\|_{L^2}. \end{aligned}$$

However by Lemma 5.5  $\text{supp } G_{-\infty} \subset (-\infty, 1/16]$  so

$$\|G_{-\infty}(\lambda^{-1}\cdot)\check{\text{Ai}}(\cdot - y)\|_{L^2([0, \infty))} \leq C\lambda^{1/2}\exp(-\lambda^{3/2}/24)\|F\|_{L^2}$$

Combining estimates on  $(-\infty, 0]$  and  $[0, \infty)$  yields

$$\|G_{-\infty}(\lambda^{-1}\cdot)\check{\text{Ai}}(\cdot - y)\|_{L^2} \leq C\exp(-\lambda)\|F\|_{L^2}.$$

Hence

$$\|G_{-\infty}(\mathcal{A}/\lambda)\delta_y\|_{L^2} \leq C\exp(-\lambda)\|F\|_{L^2}$$

and

$$\begin{aligned} \|G_{-\infty}(\mathcal{A}/\lambda)\delta_y\|_{L^1(I_{2\lambda})} &\leq |I_{2\lambda}|^{1/2}\|G_{-\infty}(\mathcal{A}/\lambda)\delta_y\|_{L^2} \\ &\leq C\lambda^{1/2}\exp(-\lambda)\|F\|_{L^2} \leq C'\|F\|_{L^2}. \end{aligned}$$

□

**Lemma 5.8.** *Let  $\mathcal{A}$  be the Airy operator defined by (4.17) and  $G_\infty$  be the function defined in Lemma 5.5. Then there exists a constant  $C > 0$ , such that for all  $\lambda \geq 1$*

$$\|G_\infty(\mathcal{A}/\lambda)\delta_y\|_{L^1(I_{2\lambda})} \leq C\|F\|_{L^2}.$$

*Proof.* By Lemma 4.1

$$G_\infty(\mathcal{A}/\lambda)\delta_y(\cdot) = \text{Ai} * (G_\infty(\lambda^{-1}\cdot)\check{\text{Ai}}(\cdot - y)).$$

We have  $G_\infty(x) = H(x)\eta(x)$  where  $\eta(x)$  is a smooth cutoff function supported on  $[2, \infty)$ . By Lemma 5.2

$$\sup_{x>2} (|G_\infty|(x))x^4 \leq C(\lambda^{3/2})^{-2}\|F\|_{L^2}.$$

Consequently for all  $x \in [2\lambda, \infty)$ ,

$$|G_\infty(x/\lambda)\check{\text{Ai}}(x-y)| \leq C(x/\lambda)^{-4}\lambda^{-3}(1 + |x-y|)^{-1/4}\|F\|_{L^2}.$$

and

$$\|G_\infty(\lambda^{-1}\cdot)\check{\text{Ai}}(\cdot - y)\|_{L^2} \leq C\lambda^{-5/2}\|F\|_{L^2}$$

Thus

$$\|G_\infty(\mathcal{A}/\lambda)\delta_y\|_{L^2} \leq C\lambda^{-5/2}\|F\|_{L^2}$$

and by  $\lambda \geq 1$

$$\begin{aligned} \|G_\infty(\mathcal{A}/\lambda)\delta_y\|_{L^1(I_{2\lambda})} &\leq |I_{2\lambda}|^{1/2}\|G_\infty(\mathcal{A}/\lambda)\delta_y\|_{L^2} \\ &\leq C\lambda^{1/2-5/2}\|F\|_{L^2} \leq C'\|F\|_{L^2}. \end{aligned}$$

□

For the rest of this section we set  $a = 2^{-k}\lambda$ . Parameter  $a$  shall always play the same role as in Proposition 4.4.

**Lemma 5.9.** *Let  $\mathcal{A}$  be the Airy operator and  $G_{k,m}$  be functions defined in Lemma 5.5. For any  $l > 0$  there exists  $C_l$  such that if  $a \geq \min(1, |y - ma|^{-1/2})$ , then*

$$|G_{k,m}(\mathcal{A}/\lambda)\delta_y|(x) \leq C_l \Theta_{k,m} \frac{d^{-1}}{(1 + d^{-1}|x - y|)^l} \left(1 + \frac{|y - ma|}{1 + |x - ma|}\right)^{\frac{1}{4}}$$

where  $d = \max(a^{-1/2}, |y - ma|^{1/2}/a)$  and  $\Theta_{k,m}$  are constants from Lemma 5.5. If  $a \leq \min(1, |y - ma|^{-1/2})$ , then

$$|G_{k,m}(\mathcal{A}/\lambda)\delta_y|(x) \leq C_l \Theta_{k,m} a (1 + |y - ma|)^{-1/4} (1 + |x - ma|)^{-1/4} (1 + a^2|x - ma|)^{-l}.$$

*Proof.* For any  $w \in L^2(\mathbb{R})$  we have

$$w(\mathcal{A})\delta_y(x) = w(\mathcal{A} + r)\delta_{y-r}(x - r).$$

Put  $r = ma$  and  $w(x) = G_{k,m}(\lambda^{-1}(x - ma))$ . Now in virtue of Lemma 5.5 we can apply Proposition 4.4. □

Next to investigate  $L^p$  properties of the operator  $G(\mathcal{A})$  we are going to decompose it using Lemma 5.5. For all  $0 \leq k \leq \log_2(\lambda^{3/2}/6)$  we set

$$(5.40) \quad T_k = \sum_{m \in \Delta_k} G_{k,m}(\mathcal{A}/\lambda)$$

where  $G_{k,m}$  are functions defined in Lemma 5.5.

**Lemma 5.10.** *Let  $T_k$  be operator defined by (5.40) corresponding to functions  $G$  and  $F$  described in Lemma 5.5. Assume further that  $0 < \varepsilon < s - 1/2$ . Then there exists constant  $C$  such that*

$$\|T_k\delta_y\|_{L^1} \leq C2^{-\varepsilon k}\|F\|_{H^s}$$

for all  $k$  such that  $2^k \leq \lambda$  and all  $y \in I_{2\lambda}$ .

*Proof.* We begin with decomposing the set  $\Delta_k$  in the following way. We set

$$\Omega_0 = \left\{m \in \Delta_k : \left|\frac{y}{a} - m\right| \leq 1\right\}$$

and

$$\Omega_n = \left\{m \in \Delta_k : 2^{n-1} < \left|\frac{y}{a} - m\right|^{1/2} \leq 2^n\right\}$$

for  $n > 0$ . Then we accordingly decompose operator  $T_k$  setting

$$T_k^{\Omega_n} = \sum_{m \in \Omega_n} G_{k,m}(\mathcal{A}/\lambda).$$



It is enough to prove that

$$(5.41) \quad \|T_k^{\Omega_n} \delta_y\|_{L^1} \leq C 2^{-\varepsilon k} \|F\|_{H^s}$$

for all  $n \in \mathbb{N}$ . Indeed, note that if  $y \in I_{2\lambda}$  then  $2^{2(n-1)} < |2^k y / \lambda| + m \leq 2^{k+1} + 2^{k+2} \leq 2^{k+3}$ . Hence  $\Omega_n = \emptyset$  unless  $2(n-1) \leq k+3$  so given (5.41) we get

$$\|T_k \delta_y\|_{L^1} \leq \sum_n \|T_k^{\Omega_n} \delta_y\|_{L^1} \leq C k 2^{-\varepsilon k} \|F\|_{H^s}$$

which yields the claim for any  $0 < \varepsilon' < \varepsilon$ .

To show (5.41), firstly note that if  $2^k \leq \lambda$  then  $a = 2^{-k} \lambda \geq 1 \geq \min(1, |y - ma|^{-1/2})$ . Hence by Lemma 5.9 for  $m \in \Omega_n$  and  $n \geq 0$  we have

$$(5.42) \quad |G_{k,m}(\mathcal{A}/\lambda) \delta_y| \leq C_l \Theta_{k,m} \frac{d_n^{-1}}{(1 + d_n^{-1} |x - y|)^l} \left(1 + \frac{|y - ma|}{1 + |x - ma|}\right)^{\frac{1}{4}}$$

where  $d_n = 2^{n+k/2}/\sqrt{\lambda} = 2^n/\sqrt{a}$ . For  $0 < \varepsilon' \leq s - 1/2 - \varepsilon$ , set  $r = 2^{k\varepsilon'} d_n$  and  $l > 1 + 1/(2\varepsilon')$ . Then by estimate (5.42) and Lemma 5.5

$$(5.43) \quad \begin{aligned} \int_{|x-y|>r} |T_k^{\Omega_n} \delta_y| &\leq \sum_{m \in \Omega_n} C_l \Theta_{k,m} \int_{|x-y|>r} \frac{d_n^{-1}}{(1 + d_n^{-1} |x - y|)^l} \left(1 + \frac{|y - ma|}{1 + |x - ma|}\right)^{\frac{1}{4}} dx \\ &\leq \sum_{m \in \Omega_n} C_l \Theta_{k,m} (r/d_n)^{-l+1} \leq C_l \left(2^{k+2} \sum_{0 \leq m \leq 2^{k+2}} \Theta_{k,m}^2\right)^{1/2} (2^{k\varepsilon'})^{-l+1} \\ &\leq C' 2^{k/2} 2^{-k/2} 2^{-k(s-1/2)} \|F\|_{H^s} \leq C' 2^{-k\varepsilon} \|F\|_{H^s}. \end{aligned}$$

Secondly, for  $n \geq 2$  write  $t_k^{\Omega_n}(x) = \sum_{m \in \Omega_n} G_{k,m}(\lambda^{-1}x)$ . By (5.35) and by Lemma 5.5

$$\|t_k^{\Omega_n}\|_{L^2}^2 \leq C \sum_{m \in \Omega_n} \|G_{k,m}(x/\lambda)\|_{L^2}^2 \leq C a \sum_{m \in \Omega_n} \Theta_{k,m}^2.$$

Recall that  $\text{supp } G_{k,m} \subset [(m-1)2^{-k}, (m+1)2^{-k}]$  so if  $m \in \Omega_n$  and  $|2^k x / \lambda - 2^k y / \lambda| \leq 2^{2(n-1)} - 1$  then  $t_k^{\Omega_n}(x/\lambda) \text{Ai}(x - y) = 0$ . Now if  $n \geq 2$  and  $|2^k x / \lambda - 2^k y / \lambda| \geq 2^{2(n-1)} - 1$  then  $|x - y| \geq a(2^{2(n-1)} - 1) \geq a 2^{2n}/8$ .

Hence, for  $n \geq 2$ , by Lemma 4.2 or by (2.8)

$$(5.44) \quad \begin{aligned} \|T_k^{\Omega_n} \delta_y\|_{L^2}^2 &= \int_{|x-y| \geq a 2^{2n-3}} |t_k^{\Omega_n}(x/\lambda) \text{Ai}(x - y)|^2 dx \leq C a^{-1/2} 2^{-n} \|t_k^{\Omega_n}\|_{L^2}^2 \\ &\leq C \sqrt{a} 2^{-n} \sum_{m \in \Omega_n} \Theta_{k,m}^2. \end{aligned}$$

Estimate (5.44) holds also for  $n = 0$  and  $n = 1$ . Indeed note that

$$\begin{aligned} \int |G_{k,m}(x/\lambda) \text{Ai}(y - x)|^2 dx &\leq \int_{|x-ma| \leq a} |G_{k,m}(x/\lambda)|^2 (1 + |x - y|)^{-1/2} dx \\ &\leq C \int_{|x-ma| \leq a} \Theta_{k,m}^2 (1 + |x - y|)^{-1/2} dx \leq C \Theta_{k,m}^2 a^{1/2}. \end{aligned}$$

However note that  $\#\Omega_n \leq 9$  for  $n = 0$  and  $n = 1$  so estimate (5.44) is also valid for these  $n$ .

Thirdly by (5.44)

$$\begin{aligned}
 \int_{|x-y|\leq r} |T_k^{\Omega_n} \delta_y| dx &\leq 2r^{1/2} \|T_k^{\Omega_n} \delta_y\|_{L^2} \\
 &\leq C \left( 2^{k\varepsilon'} d_n \sqrt{a} 2^{-n} \sum_{m \in \Omega_n} \Theta_{k,m}^2 \right)^{1/2} = C \left( 2^{k\varepsilon'} \sum_{m \in \Omega_n} \Theta_{k,m}^2 \right)^{1/2} \\
 (5.45) \quad &\leq C' 2^{k\varepsilon'/2} 2^{-k(s-1/2)} \|F\|_{H^s} \leq C' 2^{-k\varepsilon} \|F\|_{H^s}.
 \end{aligned}$$

Now Lemma 5.10 follows from estimates (5.43) and (5.45).  $\square$

**Lemma 5.11.** *Let  $T_k$  be operator defined by (5.40) corresponding to functions  $G$  and  $F$  described in Lemma 5.5. Assume next that  $0 < \varepsilon < s - 1/2 - 1/6$ . Then there exists  $C$  such that*

$$\|T_k \delta_y\|_{L^1} \leq C 2^{-\varepsilon k} \|F\|_{H^s}$$

for all  $k$  such that  $2^k > \lambda$ .

*Proof.* This time set

$$\Lambda_0 = \{m \in \Delta_k : |y - ma|^{-1/2} \geq a\},$$

and for  $n \geq 1$  put

$$\Lambda_n = \Omega_n \setminus \Lambda_0 = \{m \in \Delta_k : |y - ma|^{-1/2} < a \text{ and } 2^{n-1} < \left| \frac{y}{a} - m \right| \leq 2^n\}$$

where  $a = 2^{-k}\lambda$ . Then for  $n = 0, 1, 2, \dots$  we write

$$T_k^{\Lambda_n} = \sum_{m \in \Lambda_n} G_{k,m}(\mathcal{A}/\lambda).$$

If  $n \geq 1$  and  $m \in \Lambda_n$  then clearly  $a > |y - ma|^{-1/2} \geq \min(1, |y - ma|^{-1/2})$ . Hence we can use the same argument as in the proof of Lemma 5.10 to show that

$$(5.46) \quad \|T_k^{\Lambda_n} \delta_y\|_{L^1} \leq C 2^{-\varepsilon k} \|F\|_{H^s}.$$

Thus it remains to handle the operator  $T_k^{\Lambda_0}$ . First note that  $2^k > \lambda$  so  $1 > a$ . Now by definition, for  $m \in \Lambda_0$  we have  $|y - ma|^{-1/2} \geq a$  so  $a < \min(1, |y - ma|^{-1/2})$ . Consequently, by Lemma 5.9

$$\begin{aligned}
 |G_{k,m}(\lambda^{-1}\mathcal{A})\delta_y| &\leq C_l \Theta_{k,m} a (1 + |y - ma|)^{-1/4} (1 + |x - ma|)^{-1/4} (1 + a^2|x - ma|)^{-l} \\
 &\leq C'_l \Theta_{k,m} a (1 + a^2|x - y|)^{-l}.
 \end{aligned}$$

Secondly take  $0 < \varepsilon' \leq s - 1/2 - \varepsilon$ ,  $l \in \mathbb{N}$  such that  $\varepsilon'(l-1) > 1$  and set  $r = 2^{k\varepsilon'} a^{-2}$ . Then by Lemma 5.5

$$\begin{aligned}
 \int_{|x-y|>r} |T_k^{\Lambda_0} \delta_y(x)| dx &\leq \sum_{m \in \Lambda_0} C'_l \Theta_{k,m} a \int_{|x-y|>r} (1 + a^2|x - y|)^{-l} dx \\
 &\leq C \frac{(ra^2)^{-l+1}}{a} \sum_{m \in \Lambda_0} \Theta_{k,m} \leq C \frac{2^{-k(l-1)k\varepsilon'}}{a} \left( 2^{k+2} \sum_{m \in \Delta_k} \Theta_{k,m}^2 \right)^{1/2} \\
 (5.47) \quad &\leq C' 2^{k/2} \lambda^{-1} 2^{-k(s-1/2)} \|F\|_{H^s} \\
 &\leq C' 2^{-k\varepsilon} \|F\|_{H^s},
 \end{aligned}$$

where we used the fact that  $0 \leq m \leq 2^{k+2}$ , the inequality  $2^k \leq \lambda^{3/2}/6$  and the estimates

$$2^{k/2} \lambda^{-1} \leq 2^{2k/3} \lambda^{-1} \leq (\lambda^{3/2}/6)^{2/3} \lambda^{-1} = (1/6)^{2/3}.$$

Next, similarly as before set  $t_k^{\Lambda_0}(x) = \sum_{m \in \Lambda_0} G_{k,m}(x/\lambda)$ . Again by Lemma 5.5

$$\|t_k^{\Lambda_0}\|_{L^2}^2 \leq Ca \sum_{m \in \Delta_k} \Theta_{k,m}^2.$$

Hence

$$\|T_k^{\Lambda_0} \delta_y\|_{L^2}^2 = \int_{\mathbb{R}} |t_k^{\Lambda_0}(x) \text{Ai}(y-x)|^2 dx \leq Ca \sum_{m \in \Delta_k} \Theta_{k,m}^2$$

Now by the Cauchy-Schwarz inequality

$$\begin{aligned} \int_{|x-y| \leq r} |T_k^{\Lambda_0} \delta_y| &\leq 2r^{1/2} \|T_k^{\Lambda_0} \delta_y\|_{L^2} \leq C \left( 2^{k\varepsilon'} a^{-2} a \sum_{m \in \Delta_k} \Theta_{k,m}^2 \right)^{1/2} \\ &\leq C' 2^{k\varepsilon'/2} a^{-1/2} 2^{-k(s-1/2)} \|F\|_{H^s} \leq C' 2^{k\varepsilon'/2} \lambda^{-1/2} 2^{k/2} 2^{-k(s-1/2)} \|F\|_{H^s} \\ &= C' 2^{k\varepsilon'/2} \lambda^{-1/2} 2^{k/3} 2^{-k(s-1/2-1/6)} \|F\|_{H^s} \\ &\leq C' 2^{k\varepsilon'/2} \lambda^{-1/2} (c\lambda^{3/2})^{1/3} 2^{-k(s-1/2-1/6)} \|F\|_{H^s} \\ (5.48) \quad &\leq C'' 2^{-k\varepsilon} \|F\|_{H^s}. \end{aligned}$$

To get the last line we used the inequality  $2^k \leq \lambda^{3/2}/6$  and  $\varepsilon < s - 1/2 - 1/6$ .

Now Lemma 5.11 follows from estimates (5.46), (5.47) and (5.48).  $\square$

**Remark 5.12.** Note that in proofs of Lemmas 5.10 and 5.11 the assumption  $s > 3/2 = 1/2 + 1/6$  was crucial only in the last estimates of the proof of Lemma 5.11. In the rest of the argument it is sufficient to require that  $s > 1/2$ .

**Lemma 5.13.** Let  $T_k$  be operator defined by (5.40) corresponding to functions  $G$  and  $F$  described in Lemma 5.5. Suppose also that  $0 < \varepsilon < s - 1/2$  and  $2^k > \lambda$ . Then

$$\|T_k\|_{L^{4/3}(I_{2\lambda}) \rightarrow L^{4/3}(I_{2\lambda})} \leq Ck 2^{-k\varepsilon} \|F\|_{H^s}$$

*Proof.* Put  $t_k(x) = \sum_{m \in \Delta_k} G_{k,m}(x/\lambda)$ . We have

$$T_k f = \int_{\mathbb{R}} t_k(u) \varphi_u \langle f, \varphi_u \rangle$$

where  $\varphi_u(x) = \text{Ai}(x-u)$ .

Let  $\eta \in C_c^\infty(\mathbb{R})$  be a such function that  $0 \leq \eta \leq 1$ ,  $\text{supp}(\eta) \subset [-8, -2] \cup [2, 8]$  and  $\sum_j \eta(2^j x) = 1$  for all  $x \neq 0$ . For  $j > 0$  put

$$\varphi_{j,u}(x) = \eta(2^{-j}(x-u)) \varphi_u(x) \quad \text{and} \quad \varphi_{0,u} = \varphi_u - \sum_{j>0} \varphi_{j,u}.$$

Next set

$$T_{k,j} f = \int t_k(u) \varphi_u \langle f, \varphi_{j,u} \rangle du = \text{Ai} * (t_k \langle f, \varphi_{j,\cdot} \rangle).$$

First note that by (2.8)

$$\|\varphi_{j,u}\|_{L^2}^2 \leq C 2^{j/2}.$$

Secondly, by (5.35) and by Lemma 5.5

$$\int |t_k(u)|^2 du \leq a 2^{-k(2s-1)} \|F\|_{H^s}^2$$

where  $a = 2^{-k}\lambda$ . Hence

$$\begin{aligned} \|T_{k,j}f\|_{L^2}^2 &= \int |t_k(u)|^2 |\langle f, \varphi_{j,u} \rangle|^2 du \\ &\leq C 2^{j/2} \|f\|_{L^2}^2 \int |t_k(u)|^2 du \\ &\leq C a 2^{j/2-k(2s-1)} \|f\|_{L^2}^2 \|F\|_{H^s}^2. \end{aligned}$$

Thus

$$(5.49) \quad \|T_{k,j}\|_{L^2 \rightarrow L^2} \leq C (2^{j/2} a)^{1/2} 2^{-k(s-1/2)} \|F\|_{H^s}.$$

On the other hand

$$\|T_k\|_{L^2 \rightarrow L^2} \leq \|t_k\|_{L^\infty} \leq C \max_{m \in \Delta_k} \Theta_{k,m} \leq C 2^{-k\varepsilon} \|F\|_{H^s}$$

so

$$\begin{aligned} (5.50) \quad \left\| \sum_{2^j > a^{-2}} T_{k,j} \right\|_{L^2 \rightarrow L^2} &\leq \|T_k\|_{L^2 \rightarrow L^2} + \sum_{2^j \leq a^{-2}} \|T_{k,j}f\|_{L^2} \\ &\leq C(1 + \sum_{2^j \leq a^{-2}} (2^{j/2} a)^{1/2}) 2^{-k\varepsilon} \|F\|_{H^s} \\ &\leq C' 2^{-k\varepsilon} \|F\|_{H^s}. \end{aligned}$$

It follows from the definition of  $T_{k,j}$  that

$$(5.51) \quad T_{k,j} \delta_y = \int t_k(u) \varphi_u \varphi_u(y) \eta(2^{-j}(y-u)) du = t_{k,j}^y(\mathcal{A}) \delta_y$$

where  $t_{k,j}^y(u) = t_k(u) \eta(2^{-j}(y-u))$ . Thirdly note that  $2^{-j}\lambda \leq 2^k$  so if we set

$$t_{k,j}^y(\lambda u) = \sum_{m \in \Delta_k} \eta(2^{-j}(y-\lambda u)) G_{k,m}(u) = \sum_{m \in \Delta_k} G_{k,m,j}^y(u),$$

then functions  $G_{k,m,j}^y(u) = \eta(2^{-j}(y-\lambda u)) G_{k,m}(u)$  satisfy estimates (5.36), (5.37) and inclusion (5.35) from Lemma 5.5 uniformly for  $j$  and  $y$ . In addition

$$T_{k,j} \delta_y = \sum_{m \in \Delta_k} G_{k,m,j}^y(\mathcal{A}/\lambda) \delta_y.$$

We going to consider two cases:  $2^j > a^{-2}$  and  $2^j \leq a^{-2}$ . Note that if  $2^j > a^{-2}$  and  $G_{k,m,j}^y(u) \neq 0$ , then

$$|y - ma| \geq |y - \lambda u| - |\lambda u - ma| > 2^{j+1} - a > a^{-2}.$$

Hence we can repeat the argument similar to the proofs of Lemmas 5.10 and 5.11 assuming that  $m \notin \Lambda_0$ . It follows that

$$(5.52) \quad \|T_{k,j} \delta_y\|_{L^1} \leq C 2^{-k\varepsilon} \|F\|_{H^s}.$$

(Note that additional  $1/6$  was necessary only to consider the case  $m \in \Lambda_0$ .) Next we notice that  $2^j \leq |y - u|$  unless  $\eta(2^{-j}(y-u)) = 0$ . Then  $|u| \leq 4\lambda$  unless  $t_k(u) = 0$ . Hence

$2^n \leq |y - u| \leq |y| + |u| \leq 2\lambda + 4\lambda = 6\lambda \leq 62^k \leq 2^{k+3}$  unless  $t_{k,j}^y(u) = 0$ . Therefore we can assume that  $0 \leq j \leq 4k$ . Thus there are at most  $4k$  nonzero terms in the following sum so by (5.52)

$$\left\| \sum_{2^j > a^{-2}} T_{k,j} \right\|_{L^1 \rightarrow L^1} \leq Ck2^{-k\varepsilon} \|F\|_{H^s}.$$

Interpolating with estimate (5.50) yields the required estimates

$$(5.53) \quad \left\| \sum_{2^j > a^{-2}} T_{k,j} \right\|_{L^{4/3} \rightarrow L^{4/3}} \leq Ck2^{-k\varepsilon} \|F\|_{H^s}.$$

It remains to handle the case  $2^j < a^{-2}$ . Again we decompose the operator  $T_{k,j}$ , this time into two parts

$$T_{k,j}^{\Lambda_0} = \sum_{m \in \Lambda_0} G_{k,m,j}^y(\lambda^{-1}\mathcal{A}) \quad \text{and} \quad T_{k,j}^{\Lambda_0^c} = T_{k,j} - T_{k,j}^{\Lambda_0} = \sum_{m \notin \Lambda_0} G_{k,m,j}^y(\lambda^{-1}\mathcal{A})$$

Recall that functions  $G_{k,m,j}^y$  satisfies the same condition as  $G_{k,m}$  so if  $m \in \Lambda_0$  then by Lemma 5.9

$$|G_{k,m,j}^y(\lambda^{-1}\mathcal{A})\delta_y|(x) \leq C_l \Theta_{k,m} a(1 + a^2|x - y|)^{-l}.$$

With  $r = 2^{k\varepsilon'} a^{-2}$  where  $0 < \varepsilon' \leq s - 1/2 - \varepsilon$  and  $l > 1 + 1/\varepsilon'$ , like in the proof of Lemma 5.11 we have

$$(5.54) \quad \int_{|x-y|>r} |T_{k,j}^{\Lambda_0} \delta_y| \leq C' 2^{-k\varepsilon} \|F\|_{H^s}.$$

We have

$$|\varphi_{j,u}|(x) \leq C2^{-j/4}.$$

Consequently, if  $2^j \leq a^{-2}$ ,

$$\begin{aligned} \|T_{k,j}^{\Lambda_0} \delta_y\|_{L^2}^2 &= \int \left| \sum_{m \in \Lambda_0} G_{k,m}(u/\lambda) \right|^2 |\varphi_{j,u}(y)|^2 du \\ &\leq C2^{-j/2} \int \left| \sum_{m \in \Lambda_0} G_{k,m}(u/\lambda) \right|^2 du \leq C' 2^{-j/2} a2^{-k(2s-1)} \|F\|_{H^s}^2 \end{aligned}$$

and

$$\begin{aligned} \int_{|x-y|\leq r} |T_{k,j}^{\Lambda_0} \delta_y| &\leq (2r)^{1/2} \|T_{k,j}^{\Lambda_0} \delta_y\|_{L^2} \leq C2^{k\varepsilon'/2} a^{-1} 2^{-j/4} a^{1/2} 2^{-k(s-1/2)} \|F\|_{H^s} \\ &\leq C2^{-j/4} a^{-1/2} 2^{-k\varepsilon} \|F\|_{H^s} \end{aligned}$$

which, combining with estimate (5.54), implies that

$$(5.55) \quad \|T_{k,j}^{\Lambda_0}\|_{L^1 \rightarrow L^1} \leq C2^{-j/4} a^{-1/2} 2^{-k\varepsilon} \|F\|_{H^s}.$$

Remembering that  $m \notin \Lambda_0$  and repeating the argument of Lemmas 5.10 and 5.11 yield

$$(5.56) \quad \|T_{k,j}^{\Lambda_0^c}\|_{L^1 \rightarrow L^1} \leq C2^{-k\varepsilon} \|F\|_{H^s}.$$

However  $2^j \leq a^{-2}$  so combining estimates (5.55) and (5.56) shows that

$$\|T_{k,j}\|_{L^1 \rightarrow L^1} \leq C2^{-j/4} a^{-1/2} 2^{-k\varepsilon} \|F\|_{H^s}.$$

Now interpolating with estimate (5.49) gives

$$(5.57) \quad \|T_{k,j}\|_{L^{4/3} \rightarrow L^{4/3}} \leq C 2^{-k\varepsilon} \|F\|_{H^s}.$$

Now Lemma 5.13 follows from estimates (5.53) and (5.57).  $\square$

The proof of Lemma 5.13 concludes also the proof of Theorem 1.2.

## 6. NECESSARY CONDITIONS FOR BOCHNER-RIESZ SUMMABILITY AND PROOF OF THEOREM 1.3

This section is devoted to the discussion of a necessary condition of the boundedness of Bochner-Riesz means of anharmonic oscillator  $\mathcal{L} = -\frac{d^2}{dx^2} + |x|$ .

**Theorem 6.1.** *Suppose that  $\mathcal{L}$  is an anharmonic oscillator defined by formula (1.1) and that the Bochner-Riesz means of order  $\alpha$  are uniformly bounded on  $L^p$ , that is*

$$\sup_{R>0} \|\sigma_R^\alpha(\mathcal{L})\|_{p \rightarrow p} \leq C < \infty.$$

*Then it necessarily follows that*

$$\alpha \geq \max \left\{ 0, \frac{2}{3} \left| \frac{1}{2} - \frac{1}{p} \right| - \frac{1}{6} \right\}.$$

*In addition for  $p = 4$  and  $p = 4/3$  the necessary condition is  $\alpha > 0$ .*

*Proof.* We start the proof by introducing the distributions  $\chi_-^a$ , defined by

$$\chi_-^a = \frac{x_-^a}{\Gamma(a+1)},$$

where  $\Gamma$  is the Gamma function and

$$x_-^a = |x|^a \quad \text{if } x \leq 0 \quad \text{and} \quad x_-^a = 0 \quad \text{if } x > 0.$$

Then  $x_-^a$  are clearly distributions for  $\operatorname{Re} a > -1$  and we have for  $\operatorname{Re} a > 0$ ,

$$\frac{d}{dx} x_-^a = -a x_-^{a-1} \implies \frac{d}{dx} \chi_-^a = -\chi_-^{a-1}$$

which we use to extend the family of functions  $\chi_-^a$  to a family of distributions on  $\mathbb{R}$  defined for all  $a \in \mathbb{C}$ , see [20] for details. Since  $1 - \chi_-^0(x)$  is the Heaviside function, it follows that

$$(6.1) \quad \chi_-^{-k} = (-1)^k \delta_0^{(k-1)}, \quad k = 1, 2, \dots,$$

where  $\delta_0$  is the  $\delta$ -Dirac measure. Motivated by the above equality we define distribution  $\delta_-^\nu$  for all real exponents  $\nu \in \mathbb{R}$  by the formula

$$\delta_-^\nu = \chi_-^{-\nu-1}.$$

A straightforward computation shows that for all  $w, z \in \mathbb{C}$

$$\chi_-^w * \chi_-^z = \chi_-^{w+z+1}$$

where  $\chi_-^w * \chi_-^z$  is the convolution of the distributions  $\chi_-^w$  and  $\chi_-^z$ , see [20, (3.4.10)]. It follows from the above relation that

$$\delta_-^\nu * \delta_-^\mu = \delta_-^{\nu+\mu}.$$

Now if  $\text{supp } F \subset [0, \infty)$  we define the Weyl fractional derivative of  $F$  of order  $\nu$  by the formula

$$F^{(\nu)} = F * \delta_-^\nu$$

and we note that

$$F^{(\nu)} * \delta_-^{-\nu} = F * \delta_-^\nu * \delta_-^{-\nu} = F,$$

see [14, Page 308] or [11, (6.5)]. Thus

$$F(\mathcal{L}) = \frac{1}{\Gamma(\nu)} \int_0^\infty F^{(\nu)}(s)(s - \mathcal{L})_+^{\nu-1} ds = \frac{1}{\Gamma(\nu)} \int_0^\infty F^{(\nu)}(s)s^{\nu-1}\sigma_s^{\nu-1}(\mathcal{L})ds, \quad \forall \nu > 0$$

for all  $F$  supported in the positive half-line. Hence if  $\text{supp } F \subset [0, \infty)$  then

$$(6.2) \quad \|F(\mathcal{L})\|_{L^p \rightarrow L^p} \leq C \sup_{R>0} \|\sigma_R^\nu(\mathcal{L})\|_{L^p \rightarrow L^p} \int_0^\infty |F^{(\nu+1)}(s)| s^\nu ds.$$

Consider now function  $\eta \in C_c^\infty(\mathbb{R})$  such that  $\eta(0) = 1$  and  $\text{supp } \eta \subset [-\frac{\pi}{2}, \frac{\pi}{2}]$  and let  $\{\lambda_n\}$  be the set of eigenvalues of the operator  $\mathcal{L}$ . We define sequence of functions  $F_n$  by the formula

$$F_n(\lambda) = \eta(\sqrt{\lambda_{n+1}}(\lambda - \lambda_n)).$$

It follows from (2.4) that  $F_n(\lambda_m) = 1$ , if  $n = m$  and  $F_n(\lambda_m) = 0$  otherwise. Hence

$$F_n(\mathcal{L})f = \sum_{m=1}^\infty F_n(\lambda_m) \langle \phi_n, f \rangle \phi_n = \langle \phi_n, f \rangle \phi_n.$$

Thus by Lemma 2.2 and estimates (2.6) for all  $p < 2$

$$\|F_n(\mathcal{L})\|_{p \rightarrow p} = \|\phi_n\|_p \|\phi_n\|_{p'} \geq c\lambda_n^{-1/4} \lambda_n^{1/p-1/2}$$

where  $p'$  is conjugate exponent of  $p$ , that is  $1/p + 1/p' = 1$ . In addition for  $p = 4/3$ .

$$\|F_n(\mathcal{L})\|_{4/3 \rightarrow 4/3} = \|\phi_n\|_{4/3} \|\phi_n\|_4 \geq c(\ln \lambda_n)^{1/4}.$$

Next note that  $\delta_-^\nu$  is a homogenous distribution, see [20, Definition 3.2.2] so  $F_n^{(\nu+1)}(\lambda) = \lambda_{n+1}^{(\nu+1)/2} \eta^{(\nu+1)}(\sqrt{\lambda_{n+1}}(\lambda - \lambda_n))$ . Hence setting  $a = \lambda_n - \pi\lambda_{n+1}^{-1/2}/2$  and  $b = \lambda_n + \pi\lambda_{n+1}^{-1/2}/2$  one gets

$$(6.3) \quad \begin{aligned} \int_0^\infty |F_n^{(\nu+1)}(\lambda)| \lambda^\nu d\lambda &= \lambda_{n+1}^{(\nu+1)/2} \int_a^b \left| \eta^{(\nu+1)}(\sqrt{\lambda_{n+1}}(\lambda - \lambda_n)) \right| \lambda^\nu d\lambda \\ &\leq C\lambda_{n+1}^{(\nu+1)/2} \lambda_{n+1}^\nu (b - a) \leq C\lambda_n^{3\nu/2}. \end{aligned}$$

Now suppose that  $\sup_{R>0} \|\sigma_R^\alpha(\mathcal{L})\|_{p \rightarrow p} < \infty$ . Substituting  $\alpha = \nu$  in (6.2) and using estimate (6.3) show that if  $1 \leq p < 2$  then

$$c\lambda_n^{-1/4} \lambda_n^{1/p-1/2} \leq C\lambda_n^{3\alpha/2}.$$

The above estimates can hold for large  $n$  only if  $\alpha \geq -\frac{1}{2} + \frac{2}{3p}$  or  $p \leq \frac{4}{6\alpha+3}$ . A similar argument shows that for  $p = 4/3$  the necessary condition is  $\alpha > 0$ . We extend this necessary condition to all  $1 \leq p \leq \infty$  by duality. This ends the proof of Theorem 6.1.  $\square$

**Remark 6.2.** The method used here can be used to get the necessary condition of the boundedness of Bochner-Riesz means for harmonic oscillator which was proved in [41] in another way. More precisely if  $\mathcal{H} = -\frac{d^2}{dx^2} + x^2$  and

$$\sup_{R>0} \|\sigma_R^\alpha(\mathcal{H})\|_{p \rightarrow p} \leq C < \infty.$$

Then it necessarily follows that  $\alpha \geq \max \left\{ 0, \frac{2}{3} \left| \frac{1}{2} - \frac{1}{p} \right| - \frac{1}{6} \right\}$ .

*Proof.* This time consider function  $\eta \in C_c^\infty(\mathbb{R})$  such that  $\eta(0) = 1$  and  $\text{supp } \eta \subset [-1, 1]$  and set  $F_n(\lambda) = \eta(\lambda - 2n - 1)$ . Similarly as before

$$F_n(\mathcal{H})f = \sum_{m=1}^{\infty} F_n(\lambda_m) \langle f, h_n \rangle h_n = \langle f, h_n \rangle h_n$$

where this time  $h_n$  is  $n$ -th Hermit function. Clearly  $\int_0^\infty \left| F_n^{(\nu+1)}(\lambda) \right| \lambda^\nu d\lambda \sim n^\nu$ . On the other hand it follows from the standard asymptotic for the Hermit functions that  $\|h_n\|_p \|h_n\|_{p'} \geq Cn^{-\frac{1}{2} + \frac{2}{3p}}$ , see for example [41, Lemma 2.1]. Next if  $\sup_{R>0} \|\sigma_R^\alpha(\mathcal{H})\|_{p \rightarrow p} < \infty$  then

$$n^{-\frac{1}{2} + \frac{2}{3p}} \leq Cn^\alpha.$$

This yields the required necessary condition.  $\square$

For a sake of completeness we end this section with a discussion of the proof of Theorem 1.3, which at this point is an immediate consequence of Theorem 1.2 and 6.1.

*Proof of Theorem 1.3.* We proved in Theorem 6.1 that if  $\mathcal{L} = -\frac{d^2}{dx^2} + |x|$  and

$$\sup_{R>0} \|\sigma_R^\alpha(\mathcal{L})\|_{p \rightarrow p} = \sup_{t>0} \|\sigma_1^\alpha(t\mathcal{L})\|_{p \rightarrow p} \leq C < \infty.$$

Then it necessarily follows that  $\alpha \geq \max \left\{ 0, \frac{2}{3} \left| \frac{1}{2} - \frac{1}{p} \right| - \frac{1}{6} \right\}$ . Hence it remains to prove that if  $\alpha > \max \left\{ 0, \frac{2}{3} \left| \frac{1}{2} - \frac{1}{p} \right| - \frac{1}{6} \right\}$  then indeed the above estimate for the Riesz means holds. To show it take a function  $\psi \in C_c^\infty(-3/4, 3/4)$  such that  $\psi = 1$  on  $[-1/2, 1/2]$  so that

$$\sigma_1^\alpha(\lambda) = (1 - \lambda^2)_+^\alpha = (1 - \lambda^2)_+^\alpha \psi(\lambda) + (1 - \lambda^2)_+^\alpha (1 - \psi(\lambda)) = F_1^\alpha(\lambda) + F_2^\alpha(\lambda).$$

where  $F_1^\alpha(\lambda) = \sigma_1^\alpha(\lambda) \psi(\lambda)$  and  $F_2^\alpha(\lambda) = \sigma_1^\alpha(\lambda) (1 - \psi(\lambda))$ . Now it is enough to show that if  $\alpha > \max \left\{ 0, \frac{2}{3} \left| \frac{1}{2} - \frac{1}{p} \right| - \frac{1}{6} \right\}$  then

$$\sup_{t>0} \|F_1^\alpha(t\mathcal{L})\|_{p \rightarrow p} \leq C < \infty \quad \text{and} \quad \sup_{t>0} \|F_2^\alpha(t\mathcal{L})\|_{p \rightarrow p} \leq C < \infty.$$

Note that  $\text{supp } F_2^\alpha \subset [1/2, 1]$  and if  $\alpha + 1/2 > s$  then  $\sigma_1^\alpha \in H^s$  and  $F_2^\alpha \in H^s$ . Now the required estimate for  $\|F_2^\alpha(t\mathcal{L})\|_{p \rightarrow p}$  follows directly from Theorem 1.2. On the other hand it is not difficult to note that  $F_1^\alpha \in C_c^\infty(-3/4, 3/4)$  and required estimates for  $\|F_1^\alpha(t\mathcal{L})\|_{p \rightarrow p}$  follows from Proposition 3.8.  $\square$

In fact estimates for  $F_1^\alpha$  do not required the sharp result and follows from standard spectral multipliers theorems. Indeed it follows from the FeynmanKac formula the corresponding heat kernel satisfies Gaussian upper bounds. Now the required estimate for  $\|F_1^\alpha(t\mathcal{L})\|_{p \rightarrow p}$  follows for example from [11, Theorem 3.1 and Remark 1, page 451] or [5, Theorem 3.1].



## 7. CONCLUDING REMARKS

Next we will discuss a singular integral version corresponding to Theorem 1.2. That is we extend compactly (dyadically) supported spectral multipliers to singular integral version similar as in Fourier multipliers of Mikhlin-Hörmander type. The result can be stated in the following way.

**Theorem 7.1.** *Suppose that  $\mathcal{L}$  is an anharmonic oscillator defined by (1.1). Assume next that  $1 < p < \infty$ ,  $s > \max\{\frac{1}{2}, \frac{2}{3}|\frac{1}{2} - \frac{1}{p}| + \frac{1}{3}\}$  and the bounded Borel function  $F$  satisfies*

$$\sup_{t>0} \|\eta \delta_t F\|_{H^s} < \infty$$

where  $\eta \in C_c^\infty(1/4, 4)$  is a fixed non-trivial auxiliary function.

Then the operators  $F(\mathcal{L})$  are bounded on space  $L^p(\mathbb{R})$  and the following estimate

$$\|F(\mathcal{L})\|_{p \rightarrow p} \leq C \sup_{t>0} \|\eta \delta_t F\|_{H^s} < \infty.$$

holds for the multiplier  $F(\mathcal{L})$ .

*Proof.* The result and the above estimates follow directly from [32, Theorem 3.3].  $\square$

The following statement is an obvious consequence of Theorem 1.2. We state this results here to explain better the relation between Theorem 1.2 and the Bochner-Riesz summability result that is Theorem 1.3.

**Corollary 7.2.** *Suppose that  $\mathcal{L}$  is an anharmonic oscillator defined by (1.1) and that  $\text{supp } F \subset [1/2, 1]$ . Assume next that  $1 \leq p \leq \infty$ ,  $s > \max\{1, \frac{2}{3}|\frac{1}{2} - \frac{1}{p}| + \frac{1}{3} + \frac{1}{2}\}$  and that  $F \in W_s^1$ .*

Then the operators  $F(t\mathcal{L})$  are uniformly bounded on space  $L^p(\mathbb{R})$  and

$$\sup_{t>0} \|F(t\mathcal{L})\|_{p \rightarrow p} \leq C \|F\|_{W_s^1}.$$

*Proof.* The results is straightforward consequence of Theorem 1.2 and the fact that  $W_{s_1}^1 \subset W_{s_2}^2 = H^{s_2}$  for all  $s_1, s_2$  such that  $s_1 > s_2 + \frac{1}{2}$ .  $\square$

**Remark 7.3.** *Note that one can use estimate (6.2) to show that Corollary 7.2 follows also from Theorem 1.3. Then it is also not difficult to notice that using an argument similar as in proof of Theorem 1.3 above that this theorem also follows from the above corollary. That is Corollary 7.2 and Theorem 1.3 are equivalent statements of the same result. One can also formulate singular integral version of Corollary 7.2 similarly as in Theorem 7.1 above.*

One can use the same argument as in paragraph above to formulate Proposition 1.1 in an equivalent way using the same terms as in Corollary 7.2 with operator  $\mathcal{L}$  replaced by  $\mathcal{H}$ . This leads to a question if we could replaced the operator  $\mathcal{L}$  by  $\mathcal{H}$  also in the formulation of Theorem 1.2. The answer to this question is likely to be positive but as the format of Theorem 1.2 is essentially stronger than the statement of Corollary 7.2. Therefore the proof of such statement requires new techniques and the argument used in [41] can not be adapted to yields the version of Theorem 1.2 for  $\mathcal{H}$  without significant changes. Most likely such proof would require completely new approach. We leave this question open for future studies.

On the other hand the result obtained in [3] guarantees convergence of the Bochner-Riesz mean of order 0 that is simply the convergence of eigenfunction expansion in  $L^p(\mathbb{R})$  spaces

for all  $4/3 < p < 4$  whereas Theorem 1.3 requires the strictly positive order to assure convergence for operator  $\mathcal{L}$  in this range. It is likely that the main result of [3] holds also for  $\mathcal{L}$  but this again would require a completely new approach and we again leave this question open for future studies.

If one consider spaces  $L^p$  for  $4/3 < p < 4$  then Theorem 1.2 gives essentially stronger estimates than Proposition 3.8. Also when applied to  $L^1$  Theorem 1.2 is significantly deeper and more interesting than Proposition 3.8. Note however, that none of the spaces  $W_{2/3}^2$  and  $W_{1/2}^4$  contains the other so formally speaking these two results are of independent interest and none follows from the other. One could ask whether to assure boundedness of the multiplier  $F(\mathcal{L})$  on  $L^1$  it is enough to assume that  $F$  is in the space  $W_s^3$  for some  $s > 1/2$ . A positive answer to this problem would imply on the level of  $L^1$  both estimates from Theorem 1.2 and Proposition 3.8. It is likely though that the answer to this question is negative but we will not study this issue here. We point out however that the consideration of imaginary powers  $\mathcal{L}^{is}$  shows the  $1/2$  is the minimal possible order of differentiability for spectral multipliers in the dimension one, see [31, Theorem 1]. Then if estimates would hold with the norm of  $W_{1/2}^p$  norm of  $F$  then necessarily  $p > 3$ . Otherwise such estimates would imply convergence of Bochner-Riesz means of order smaller than  $1/6$  and this would contradict Theorem 6.1. As we mentioned above we expect that even the norm  $W_{1/2+\epsilon}^{3+\epsilon}$  of  $F$  for some very small positive  $\epsilon$  is still not enough to ensure  $L^1$  boundedness of the multiplier  $F(t\mathcal{L})$ .

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